Syracuse University
College of Engineering
& Computer Science

Linearithmic Clean-up for Vector-Symbolic Key-Value Memory with Kroneker Rotation Products

(Accepted by NeSy 2025)

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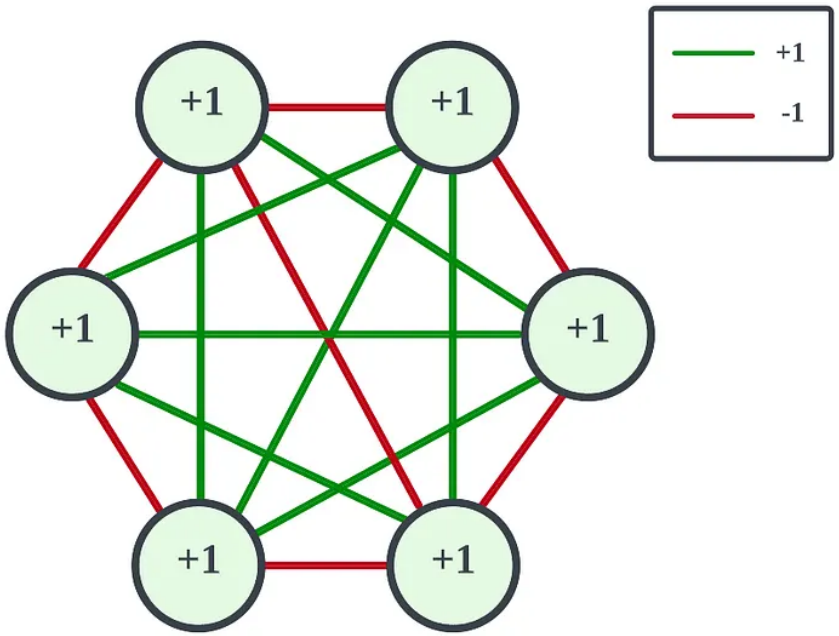
[1]Syracuse University [2]Air Force Research Laboratory

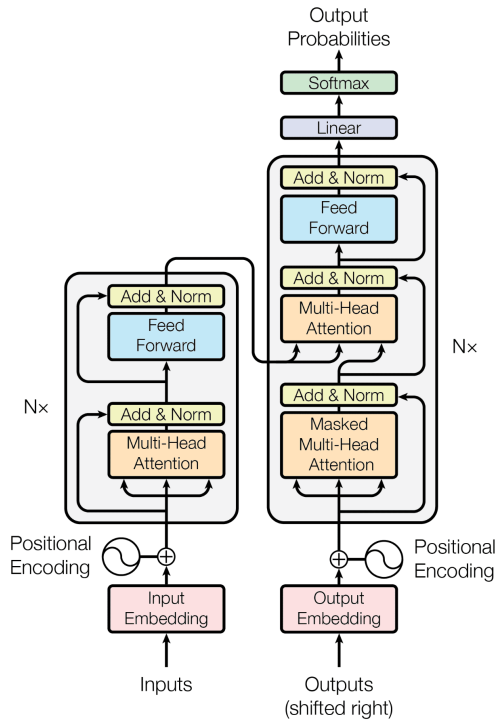
Differentiable Data Structures and Memory



Model 1

Hopfield Network [1], figure from here

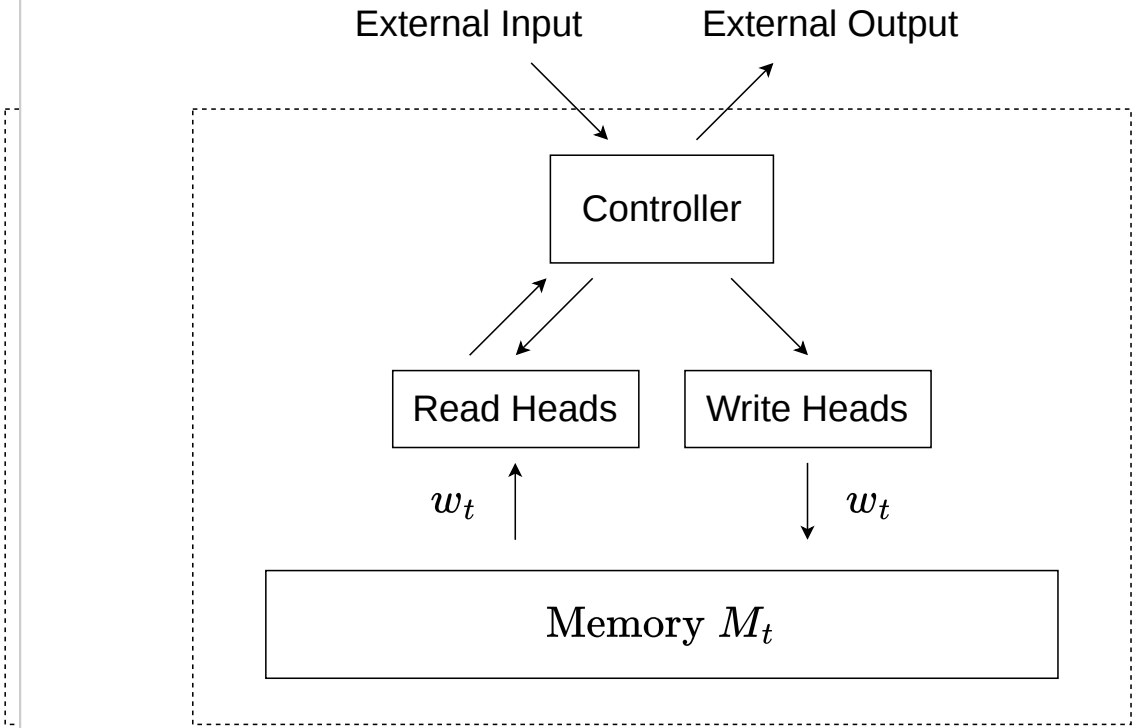
Differentiable Data Structures and Memory



Model 2

Transformer [2], figure from here

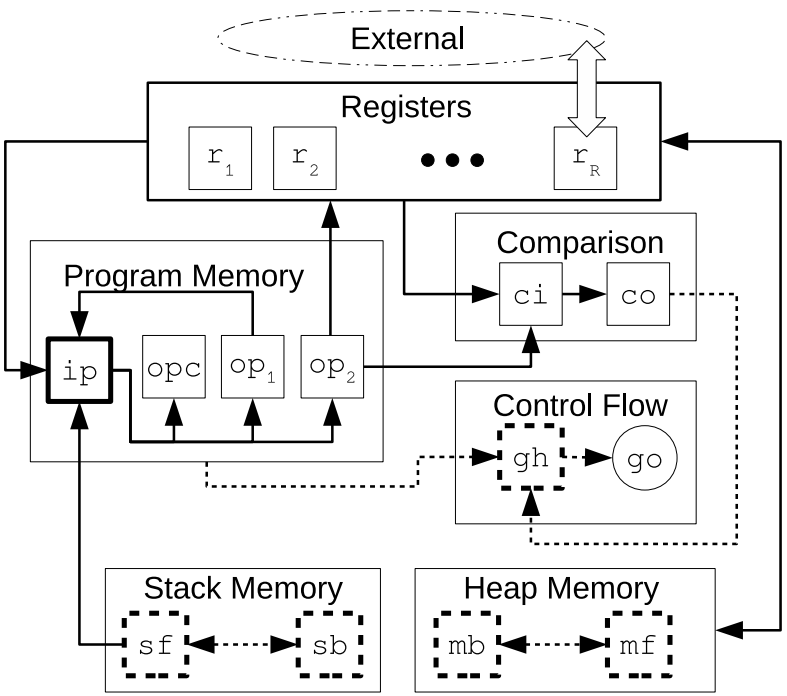
Differentiable Data Structures and Memory



Model 3

Neural Turing Machine [3]

Differentiable Data Structures and Memory



Model 4

Neural Virtual Machine [4]

Dive into NVM's Memory

Memory of NVM

Memory: $M \in \mathbb{R}^{N \times N}$

Read: $v \leftarrow \sigma(Ma)$

Write: $M \leftarrow M - \overbrace{\sigma(Ma)a^\top/N}^{\text{Erase}} + \overbrace{va^\top/N}^{\text{Write}}$

Memory of NVM

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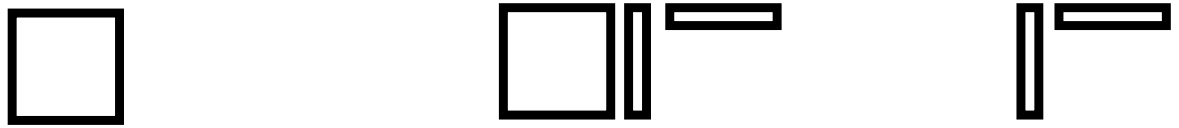

Two Problems

Capacity is N

$\Downarrow$

Single-Pass Full-Capacity Learning Rule

$\Downarrow$

ICML'24 [5] (**Does Not Exist**)

Quadratic Runtime

$\Downarrow$

Vector-**S**ymbolic **A**rchitecture (VSA) formed Memory

$\Downarrow$

NeSy'25 [6] (This Talk)

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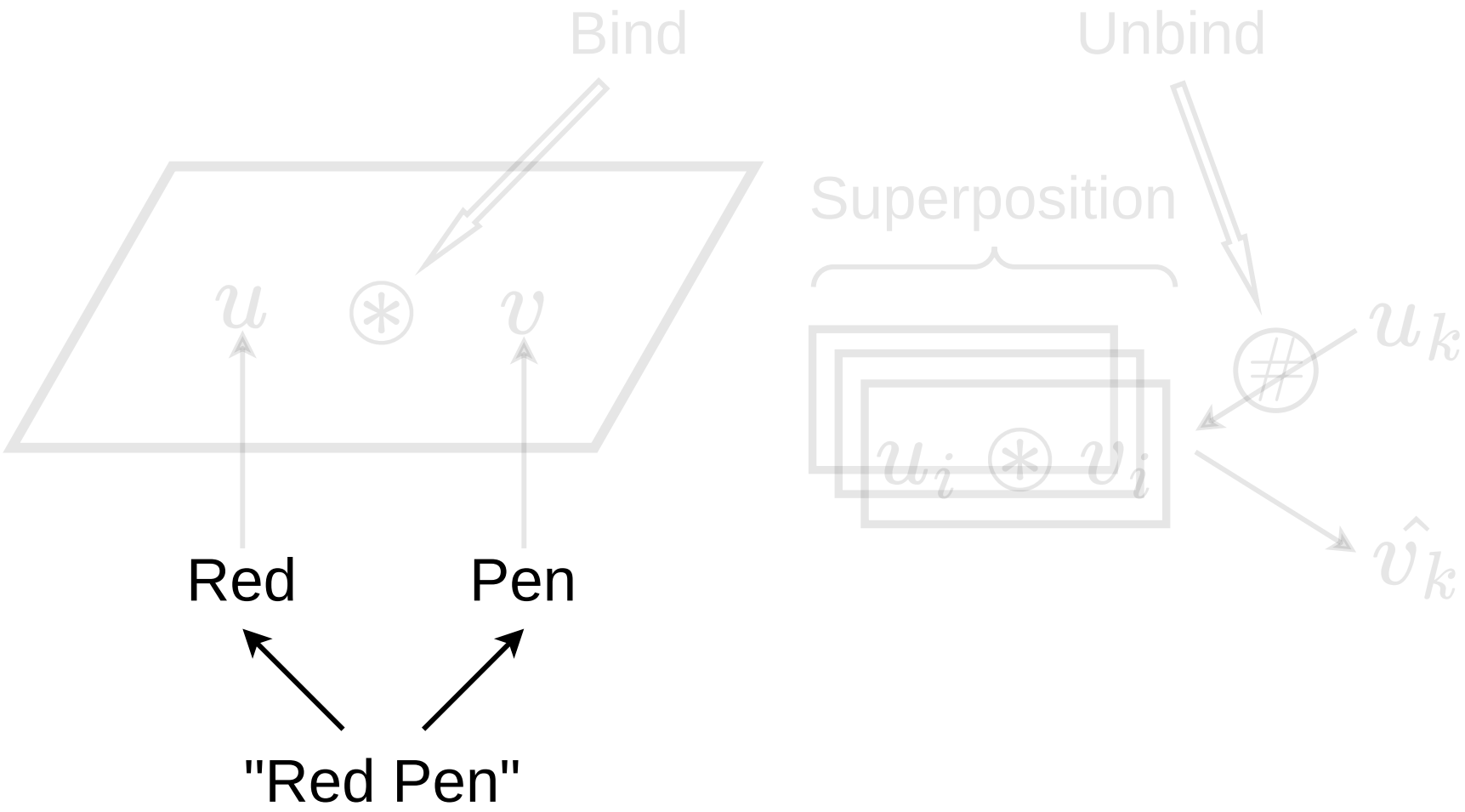
$\Downarrow$

Vector-**S**ymbolic **A**rchitecture (VSA) formed Memory

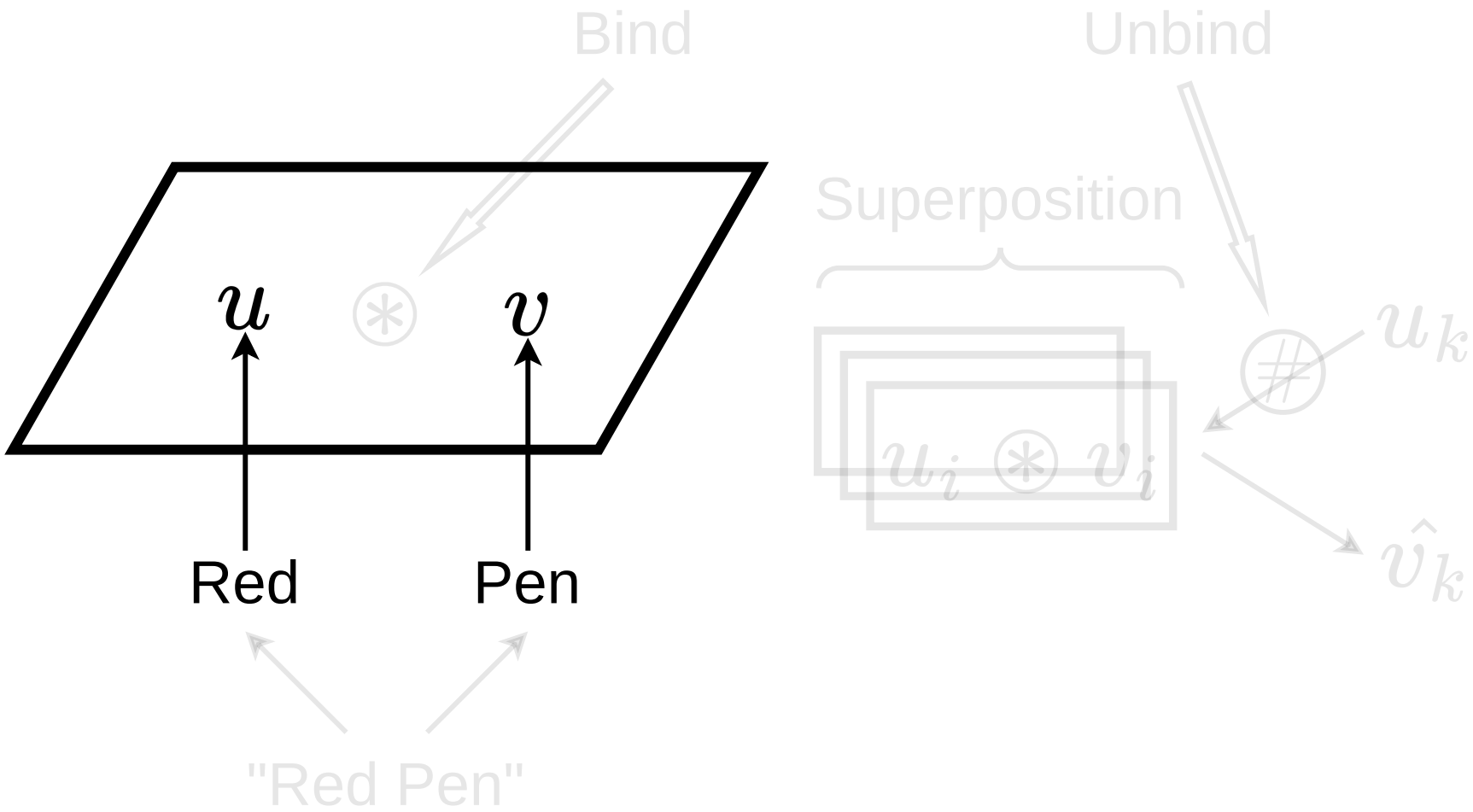
$\Downarrow$

NeSy'25 [6] (This Talk)

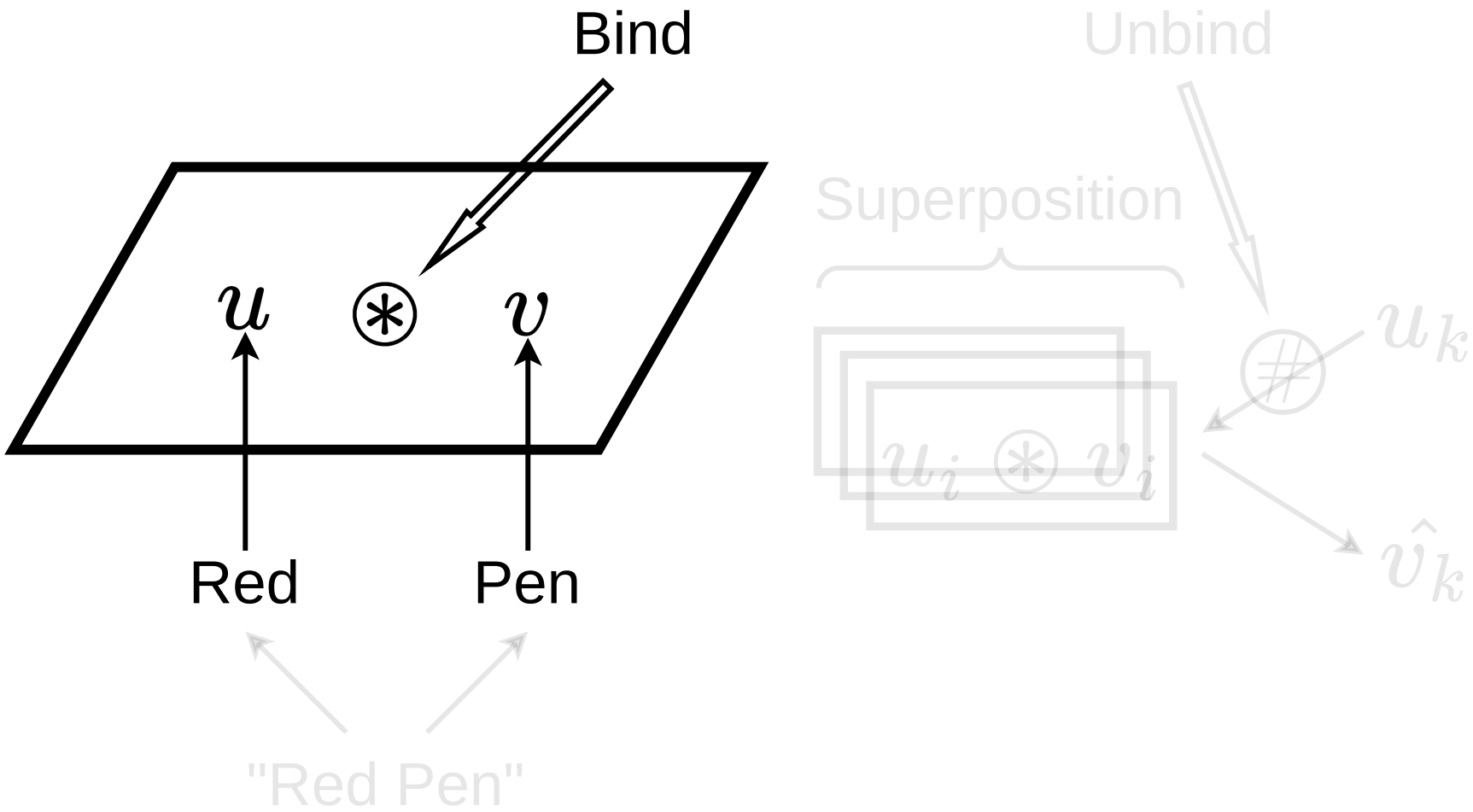
Vector Symbolic Architecture (VSA)



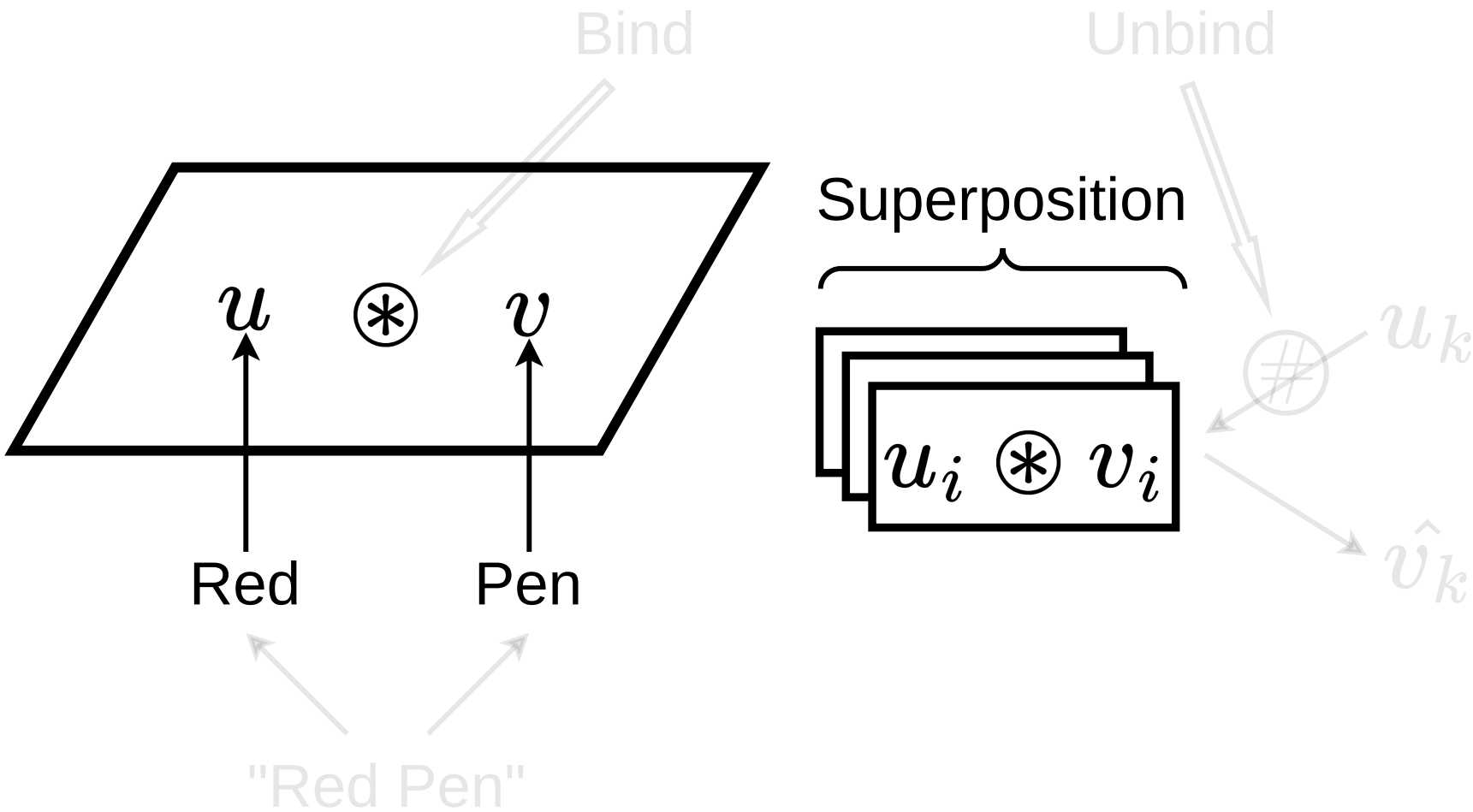
Vector Symbolic Architecture (VSA)



Vector Symbolic Architecture (VSA)



Vector Symbolic Architecture (VSA)



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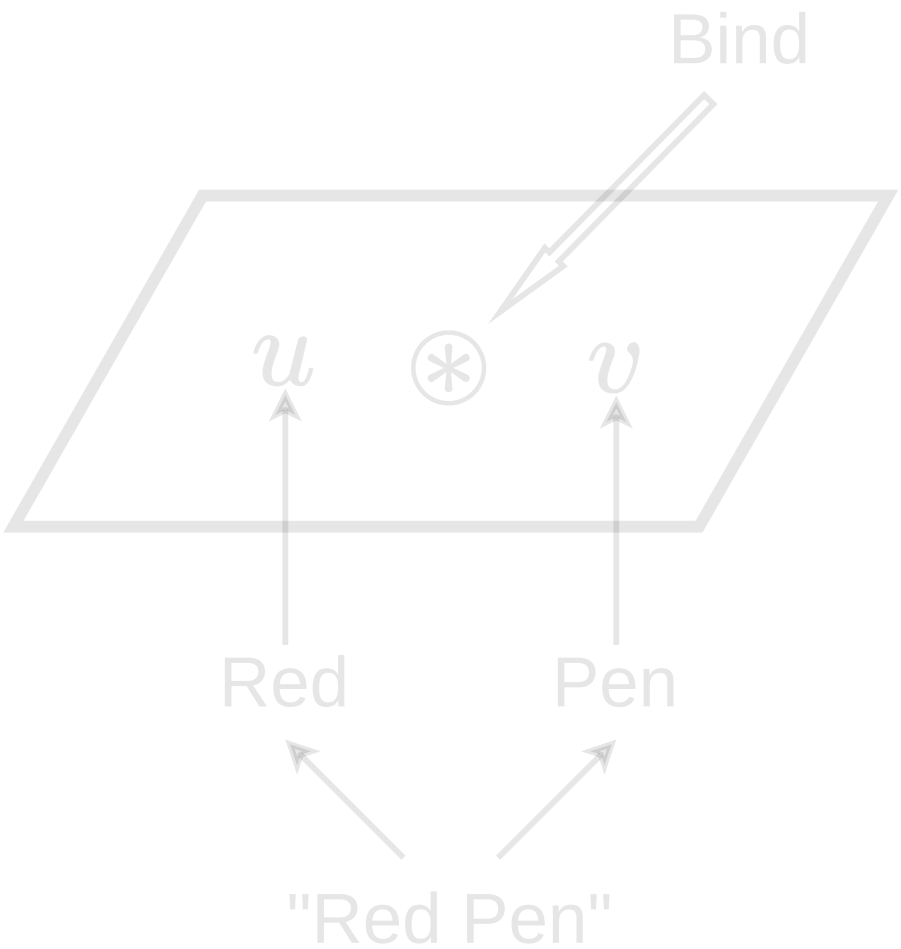
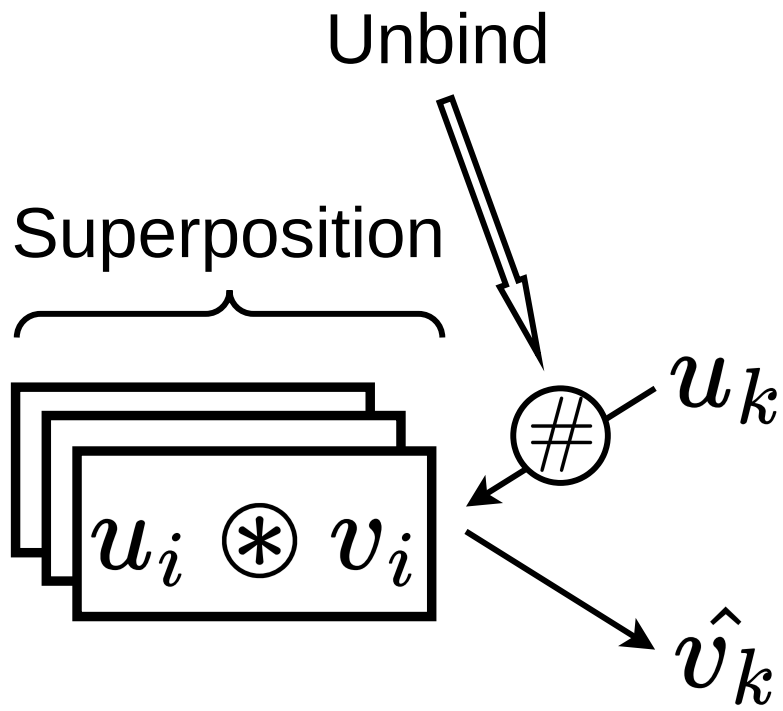
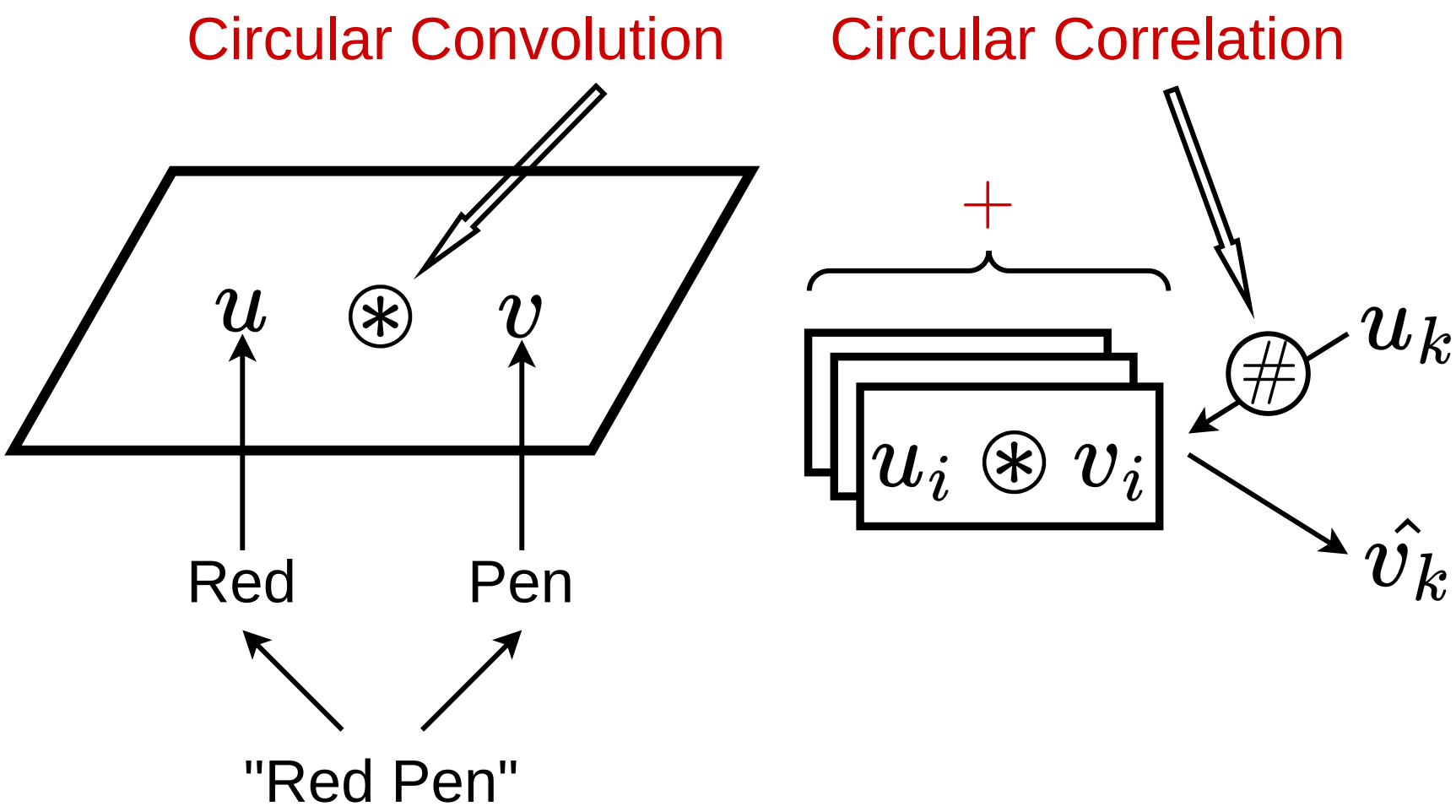
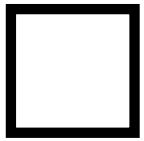


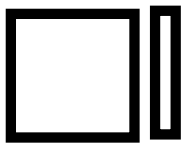


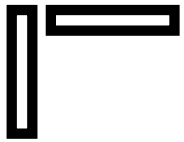
Holographic Reduced Representations (HRRs) [7]



Resulting In

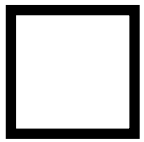
Memory: M  $\Longrightarrow$ $\mathcal{M}$

Read: $\sigma(Ma)$  $\Longrightarrow$ $\mathcal{M} \circledast a$

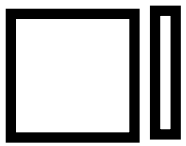
Write: $va^\top / N$  $\Longrightarrow$ $a \circledast v$

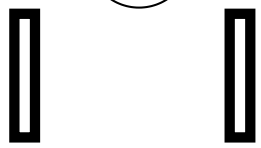
Resulting In

Memory: $M \Longrightarrow \mathcal{M}$

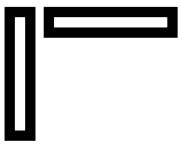


Read: $\sigma(Ma) \Longrightarrow \mathcal{M} \# a$





Write: $va^\top / N \Longrightarrow a \circledast v$



Perfect?

Issue During Reading

Memory $\mathcal{M} = \sum_i a_i \circledast v_i$

Read $\mathcal{M} \circledast a_k = v'_k$ ⟸ Contains Noise

Codebook

Clean-up $v_k^{cleaned} = \arg\max H v'_k$

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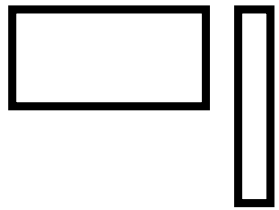

Kroneker **Ro**tation **P**roduct Codebook (`krop`)

Codebook based on **Hadamard Matrices** using Sylvester's construction and **Kroneker Rotation Products**:

$$\tilde{H}^{(0)} = [1], \quad \tilde{H}^{(k+1)} = \begin{bmatrix} \tilde{H}^{(k)}\cos(\theta_k) & \tilde{H}^{(k)}\sin(\theta_k) \\ \tilde{H}^{(k)}\sin(\theta_k) & -\tilde{H}^{(k)}\cos(\theta_k) \end{bmatrix}$$

$$\Rightarrow \quad \tilde{H}^{(K)} = \bigotimes_{k=1}^{K} \begin{bmatrix} \cos(\theta_{K-k}) & \sin(\theta_{K-k}) \\ \sin(\theta_{K-k}) & -\cos(\theta_{K-k}) \end{bmatrix}$$

where θ_k is sampled uniformly from $(0, 2\pi)$.

$\mathcal{O}(\log N)$ Storage Space

$\log N$ Storage Space

$$\overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}^{N}$$

$$\tilde{H}^{(k)} = \begin{bmatrix} \tilde{H}^{(k-1)}\cos(\theta_{k-1}) & \tilde{H}^{(k-1)}\sin(\theta_{k-1}) \\ \tilde{H}^{(k-1)}\sin(\theta_{k-1}) & -\tilde{H}^{(k-1)}\cos(\theta_{k-1}) \end{bmatrix}$$

$$\underbrace{\theta_0, \theta_1, \cdots, \theta_{k-1}}_{\mathcal{O}(\log N)}$$

log N Storage Space

$$\tilde{H}^{(k)} = \overbrace{\left[\begin{array}{cc} \tilde{H}^{(k-1)}\cos(\theta_{k-1}) & \tilde{H}^{(k-1)}\sin(\theta_{k-1}) \\ \tilde{H}^{(k-1)}\sin(\theta_{k-1}) & -\tilde{H}^{(k-1)}\cos(\theta_{k-1}) \end{array}\right]}^{N}$$

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log N Storage Space

$$\tilde{H}^{(k)} = \overbrace{\begin{bmatrix} \tilde{H}^{(k-1)}\cos(\theta_{k-1}) & \tilde{H}^{(k-1)}\sin(\theta_{k-1}) \\ \tilde{H}^{(k-1)}\sin(\theta_{k-1}) & -\tilde{H}^{(k-1)}\cos(\theta_{k-1}) \end{bmatrix}}^{N}$$

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$$\underbrace{\theta_0, \theta_1, \cdots, \theta_{k-1}}_{\mathcal{O}(\log N)}$$

Linear Time Vector Reconstruction

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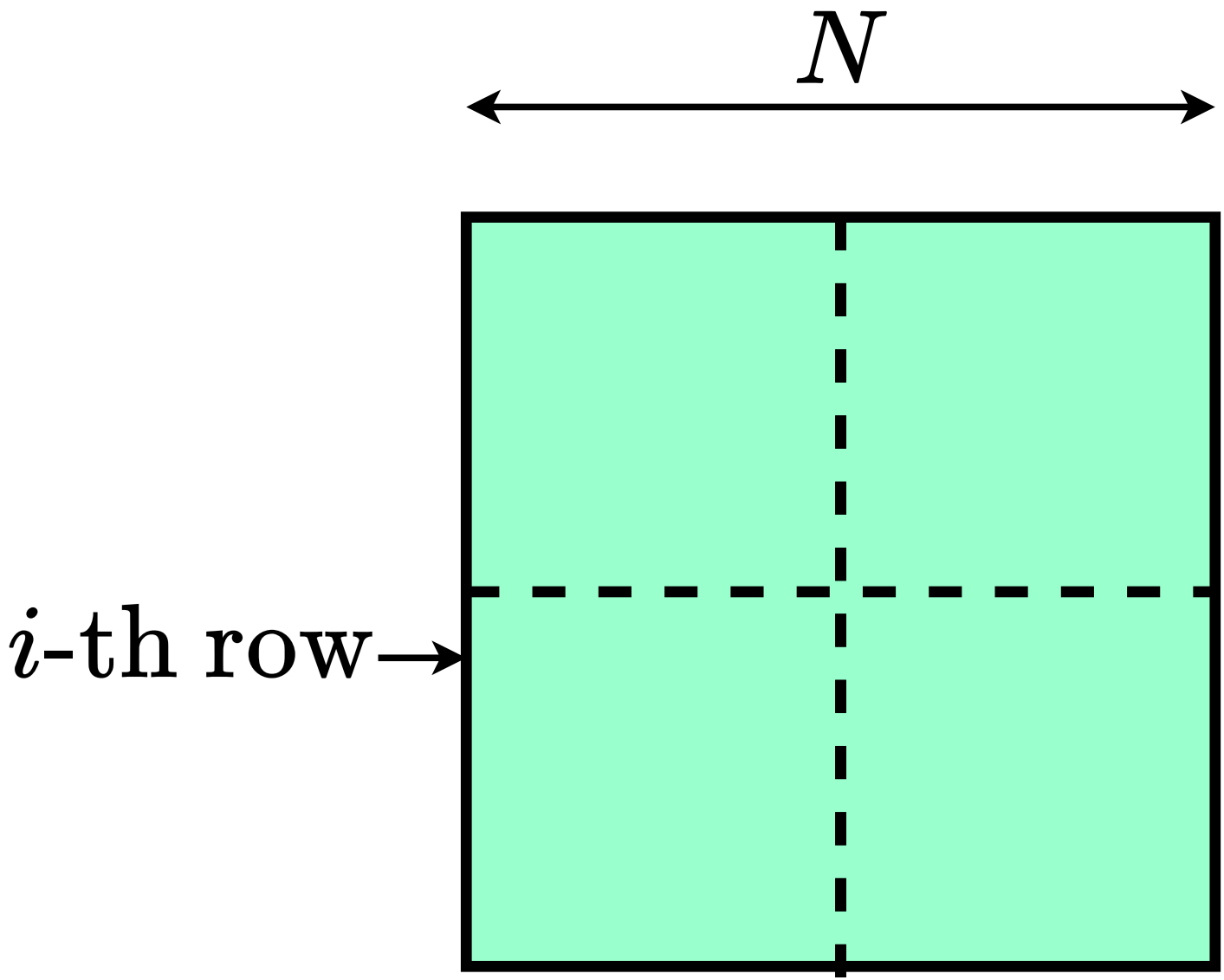
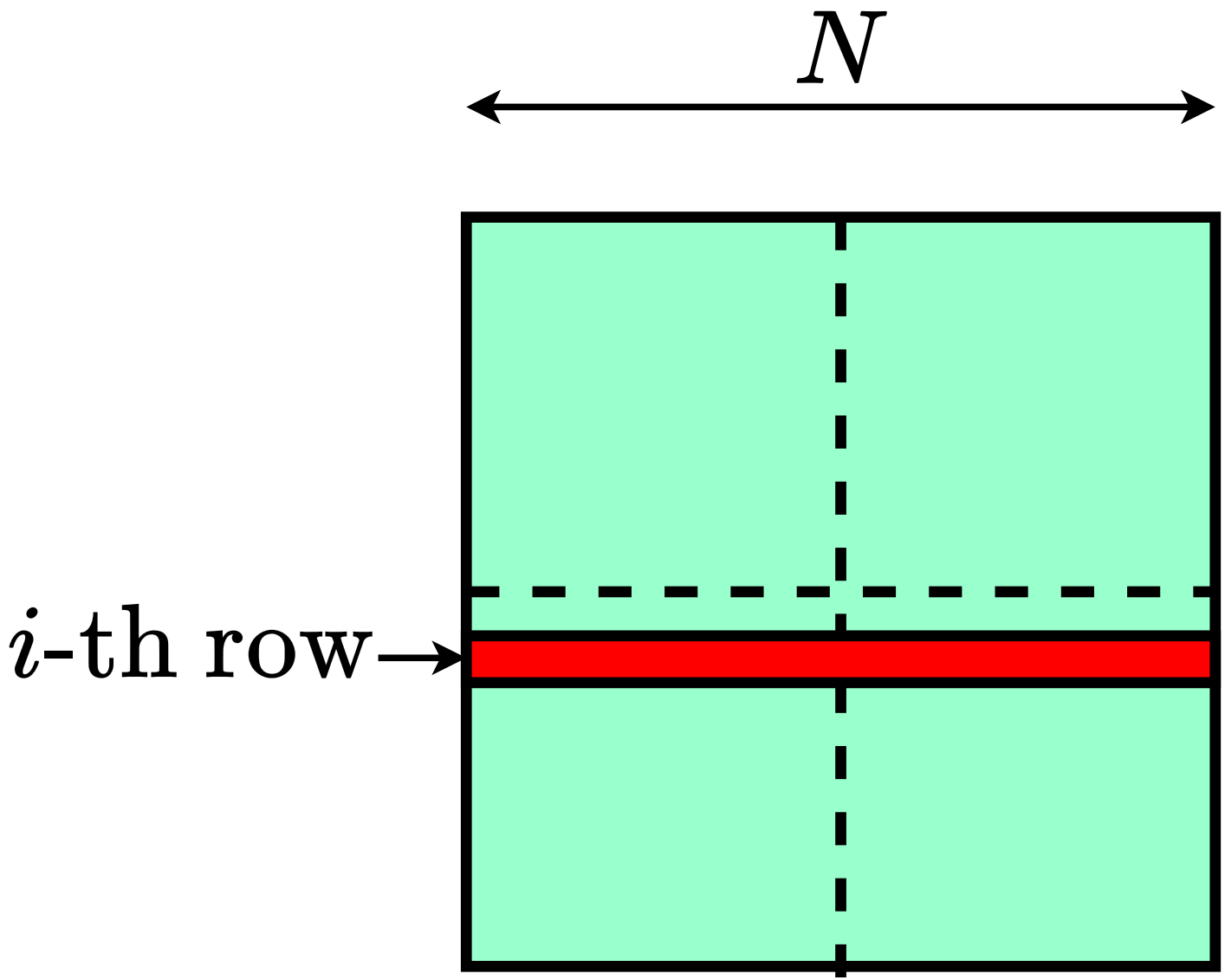


Linear Time Vector Reconstruction



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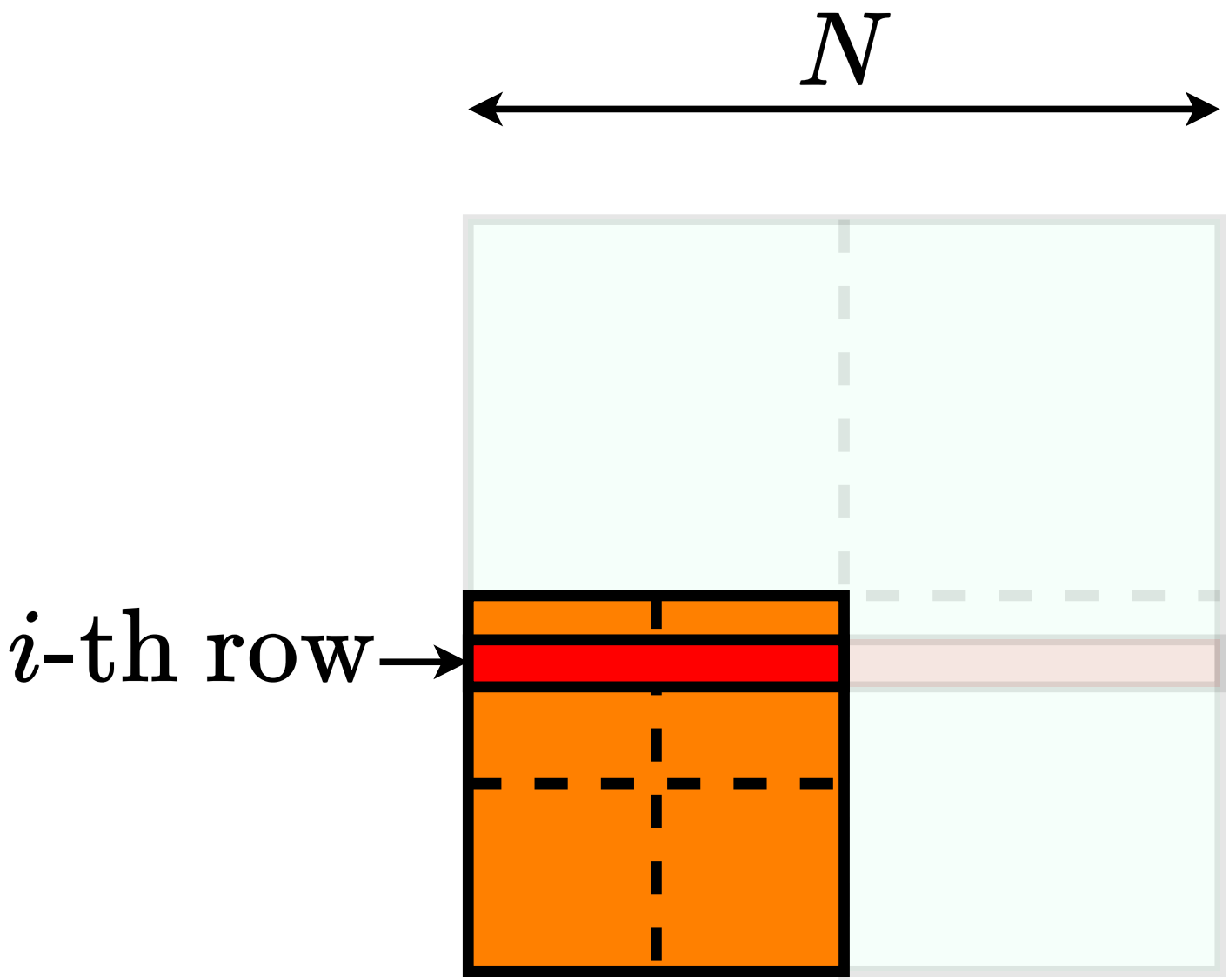

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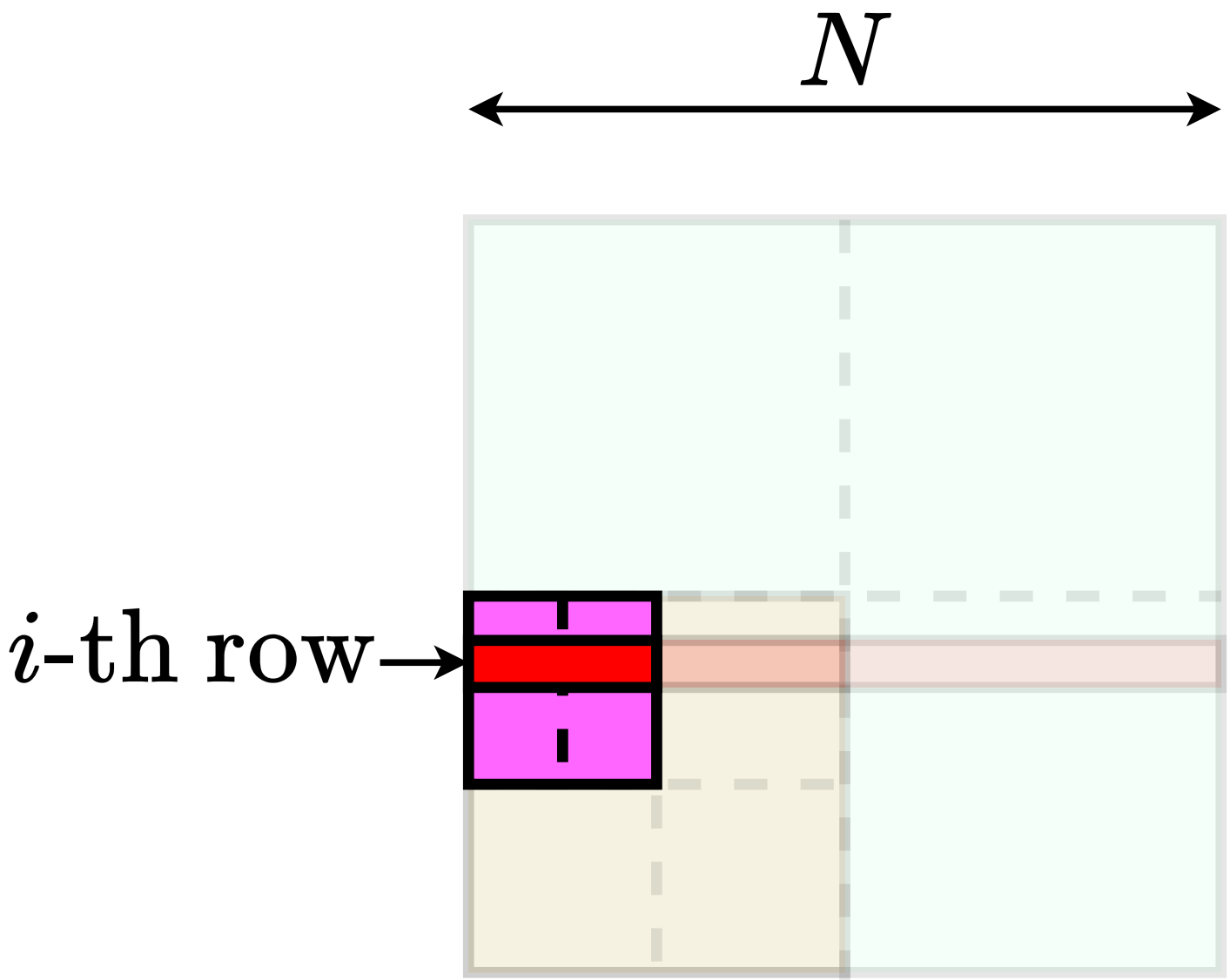
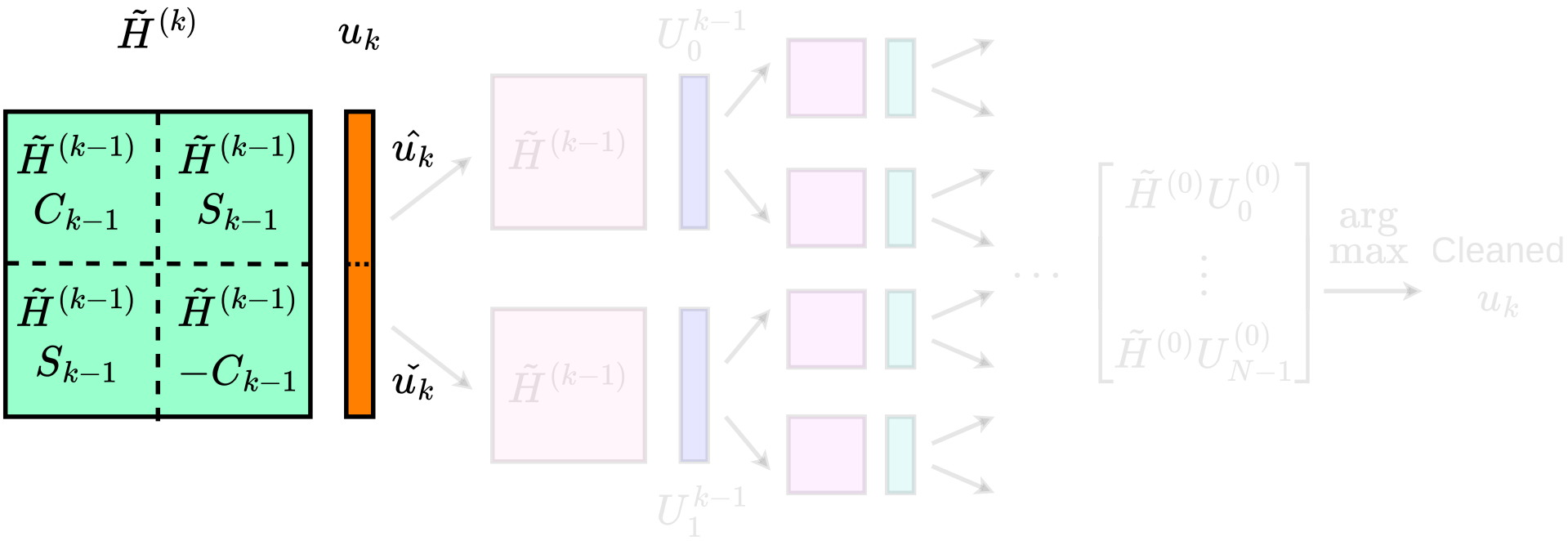
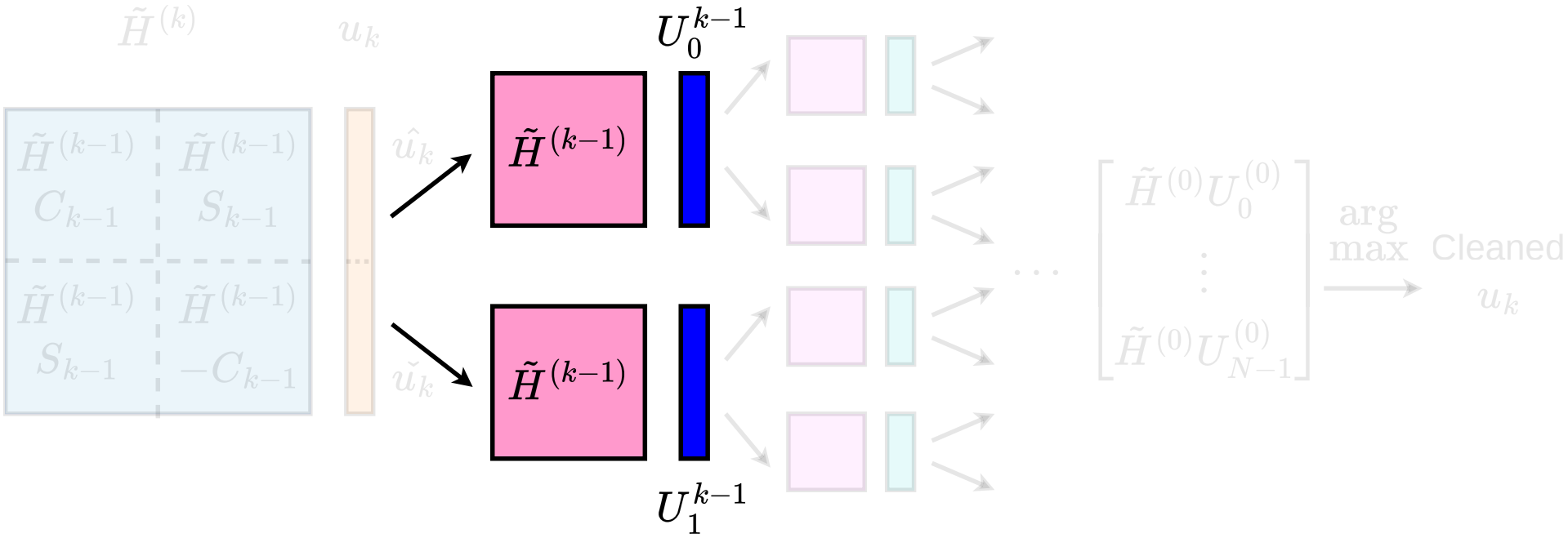


$\mathcal{O}(N \log N)$ Clean-up Process

Speeding Up the Clean-up Process



Speeding Up the Clean-up Process



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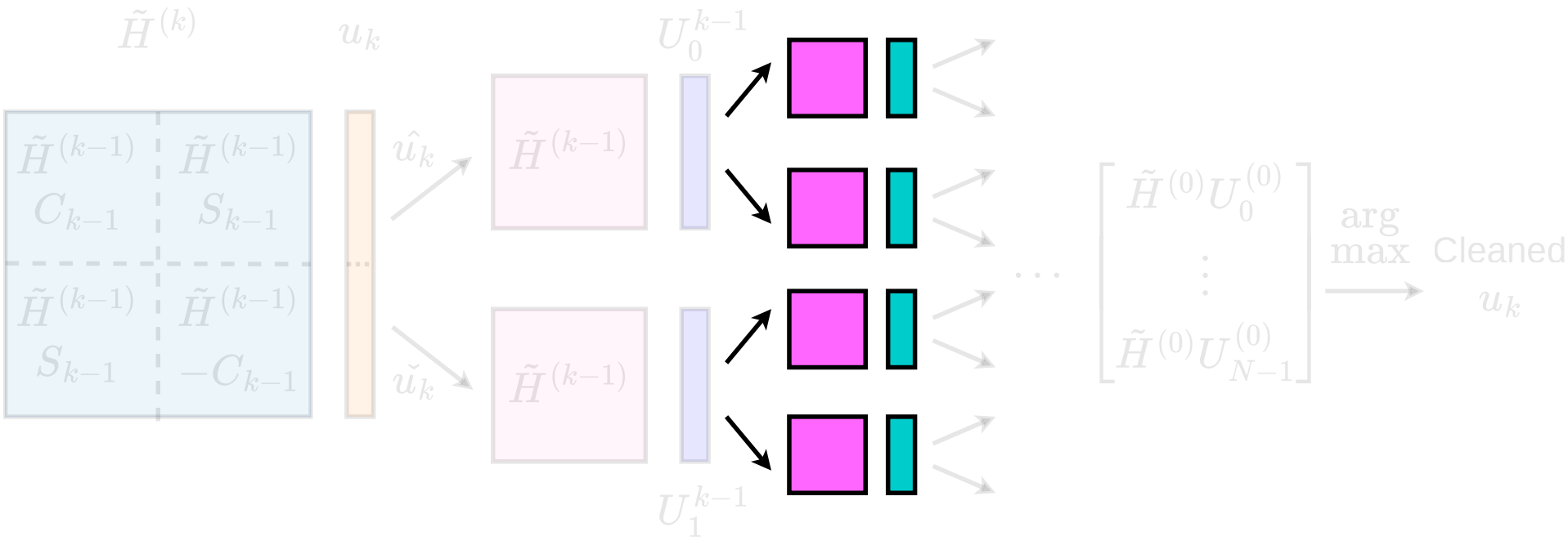

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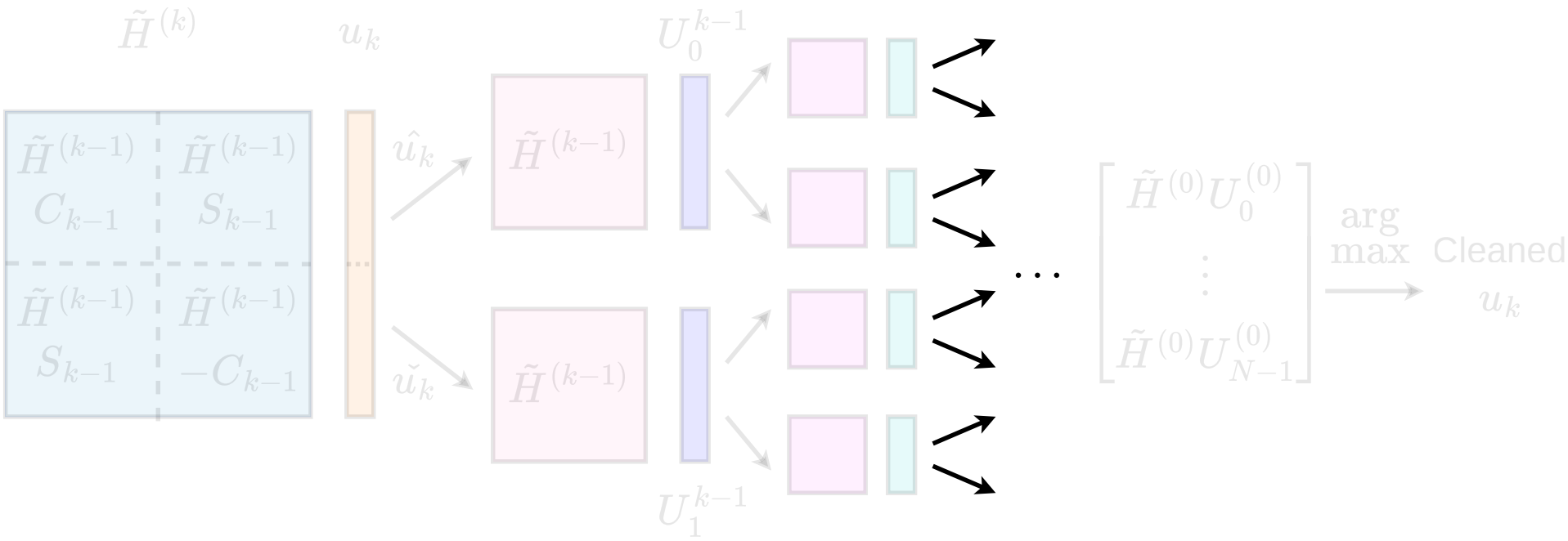
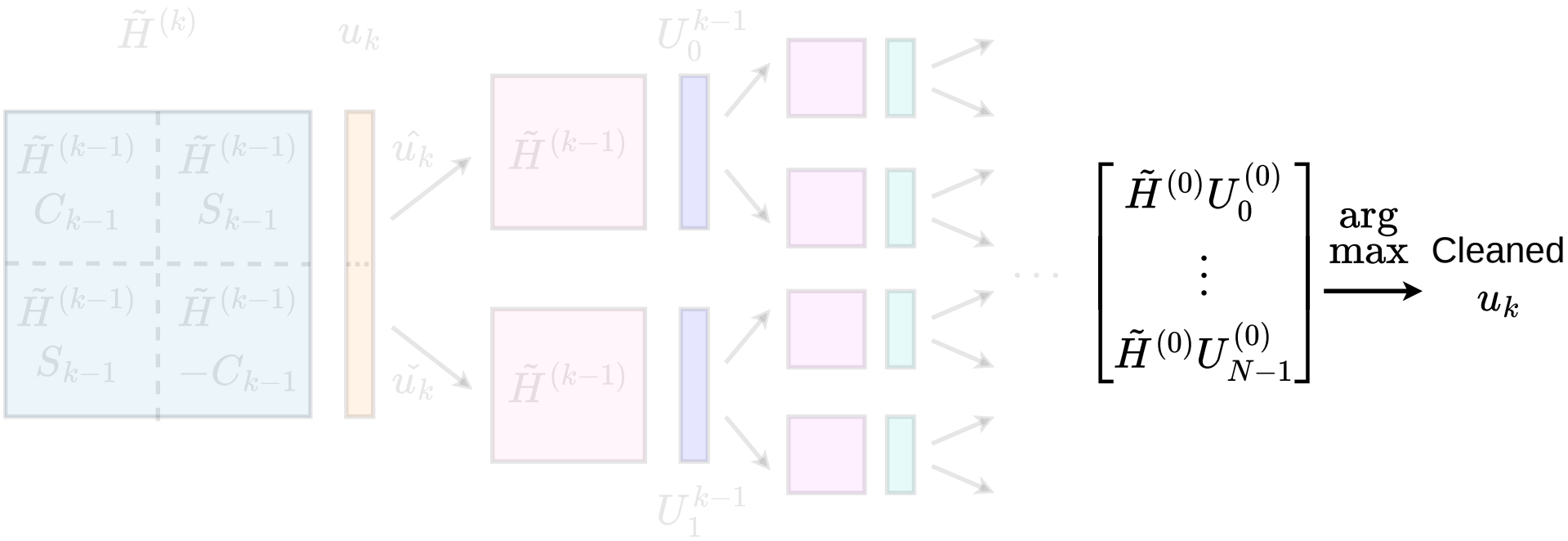


Speeding Up the Clean-up Process



Experiments

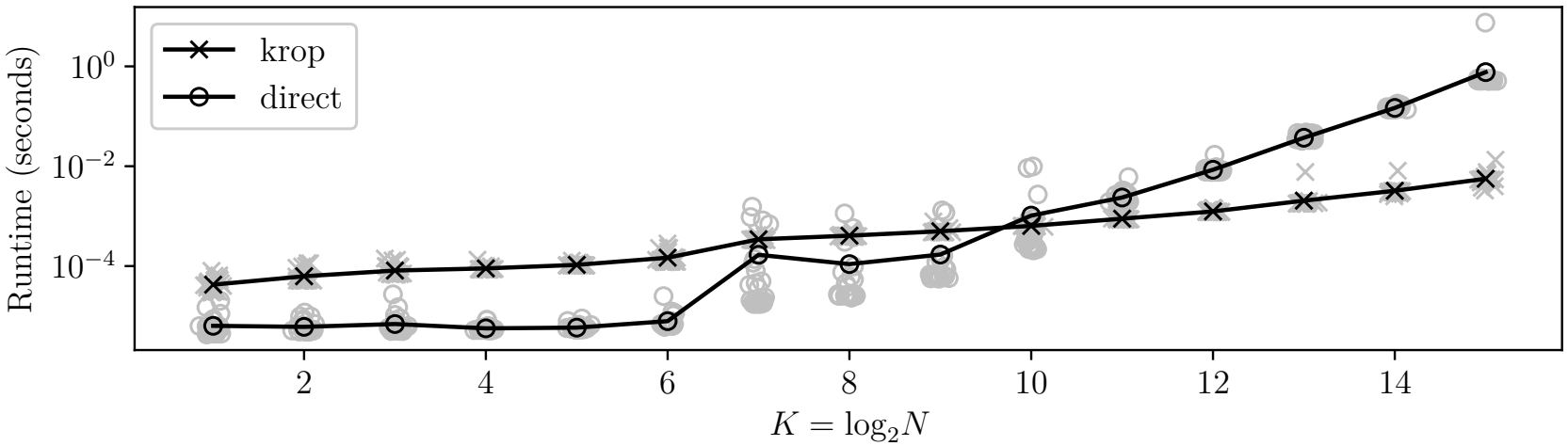
Clean-up Efficiency

Comparing clean-up runtime

How much faster?

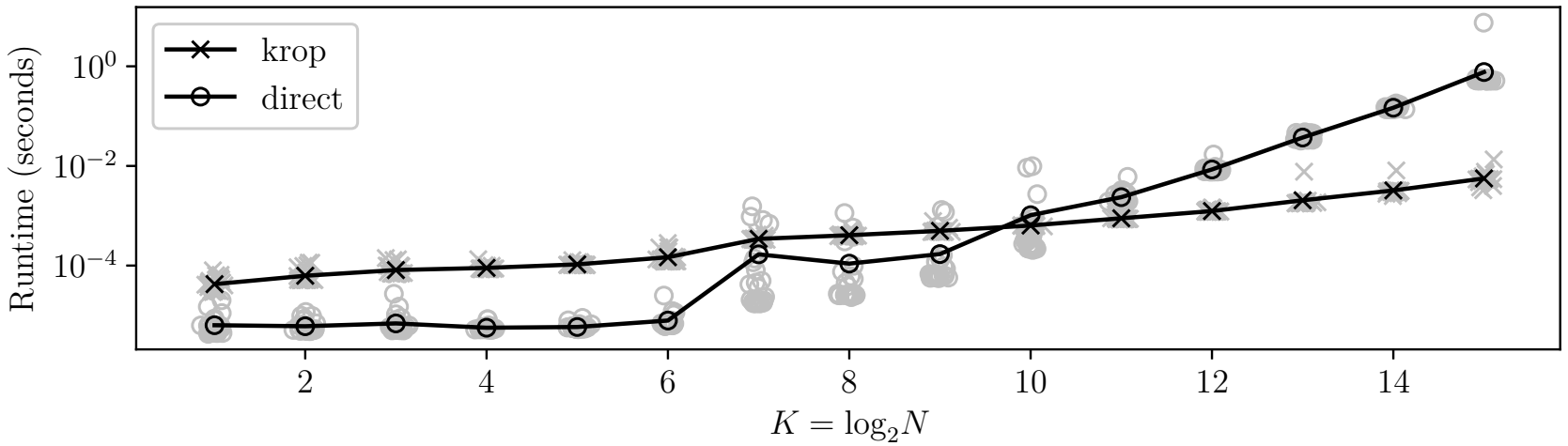
Comparing clean-up runtime

How much faster?



Comparing clean-up runtime

How much faster?

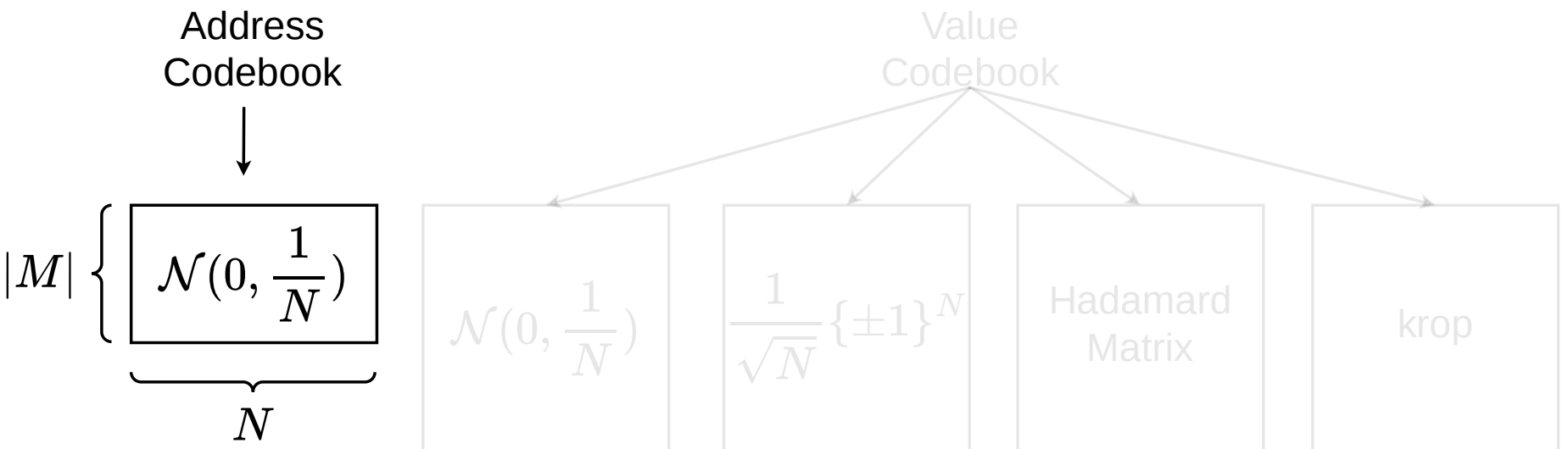


`krop` scales better.

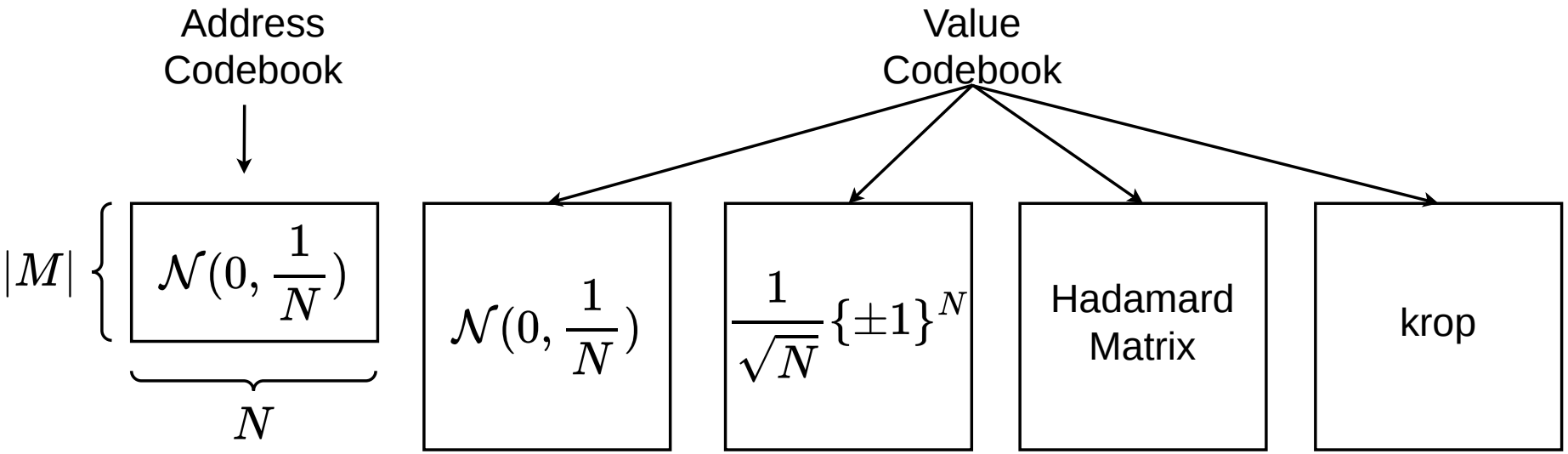
Memory Capacity Test

Is a stored item recovered correctly?

Codebooks


Address
Codebook
$|M|$
$\mathcal{N}(0, \frac{1}{N})$
N
Value
Codebook
$\mathcal{N}(0, \frac{1}{N})$
$\frac{1}{\sqrt{N}}\{\pm 1\}^N$
Hadamard
Matrix
krop

Codebooks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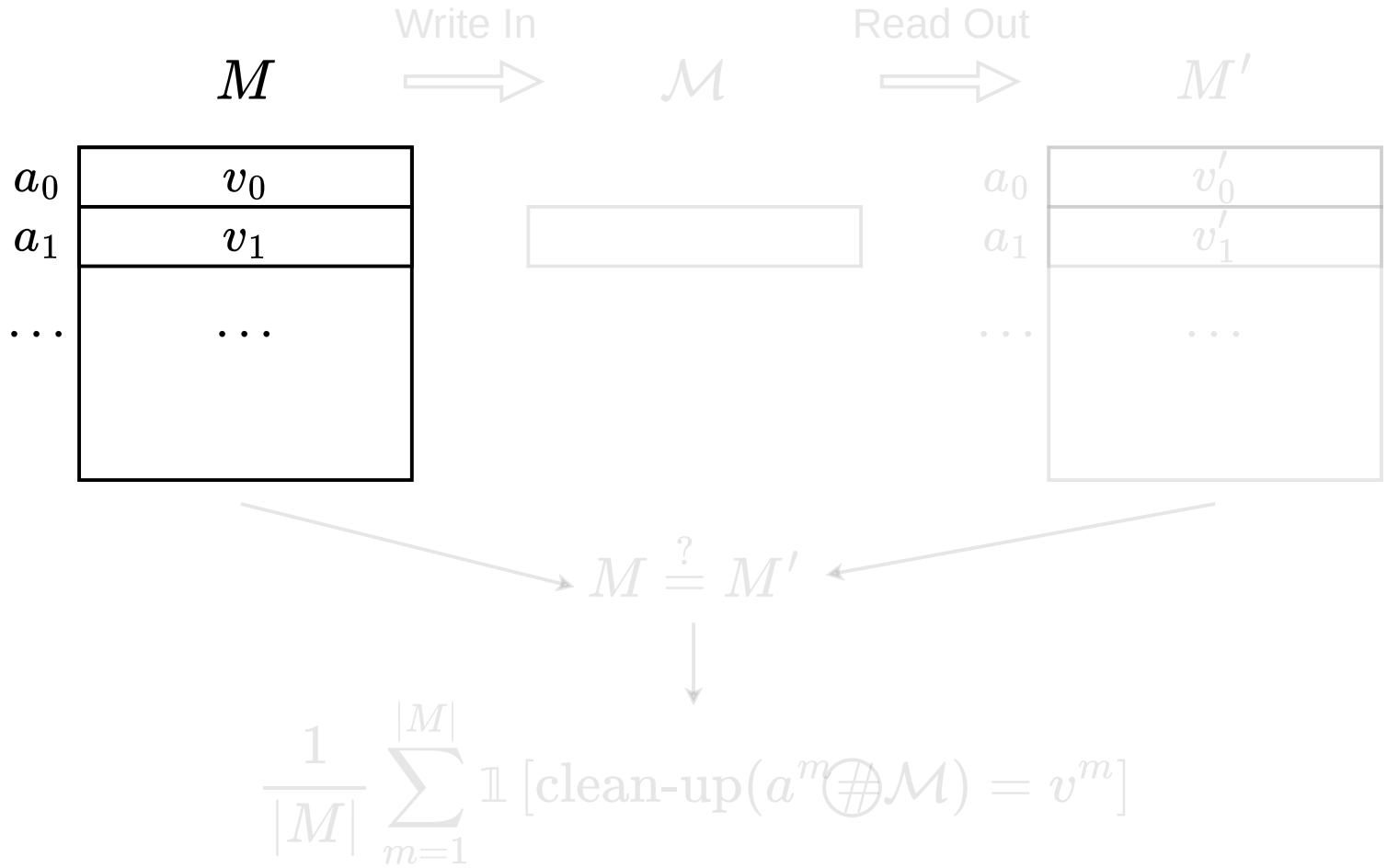
Memory Capacity Test



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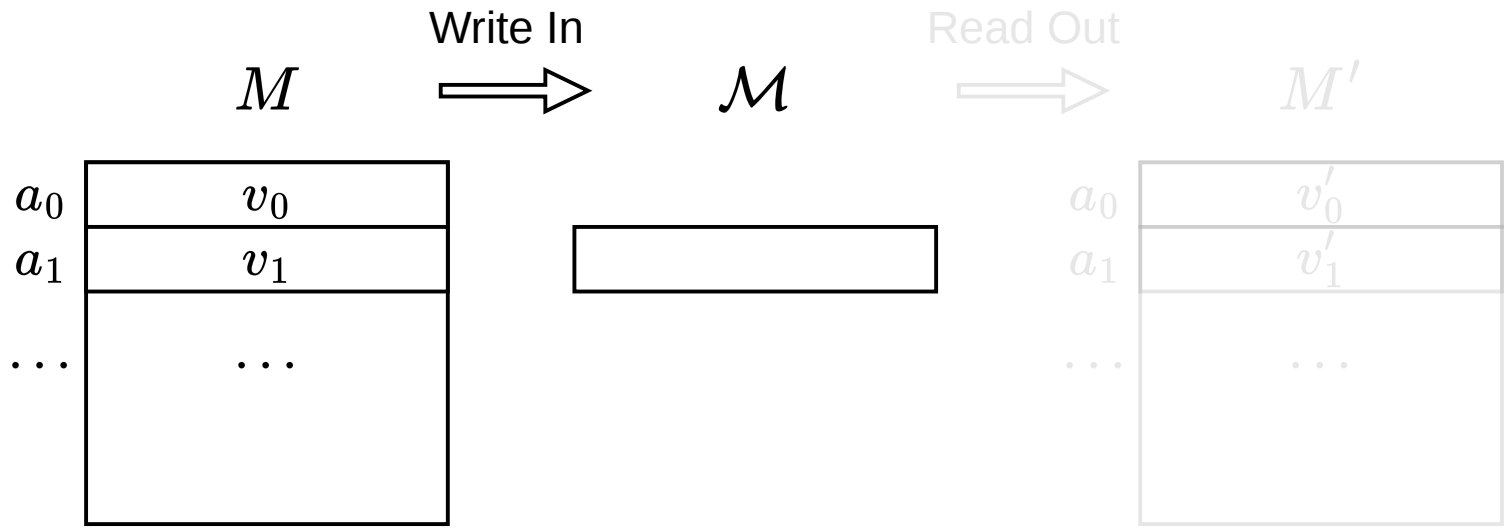
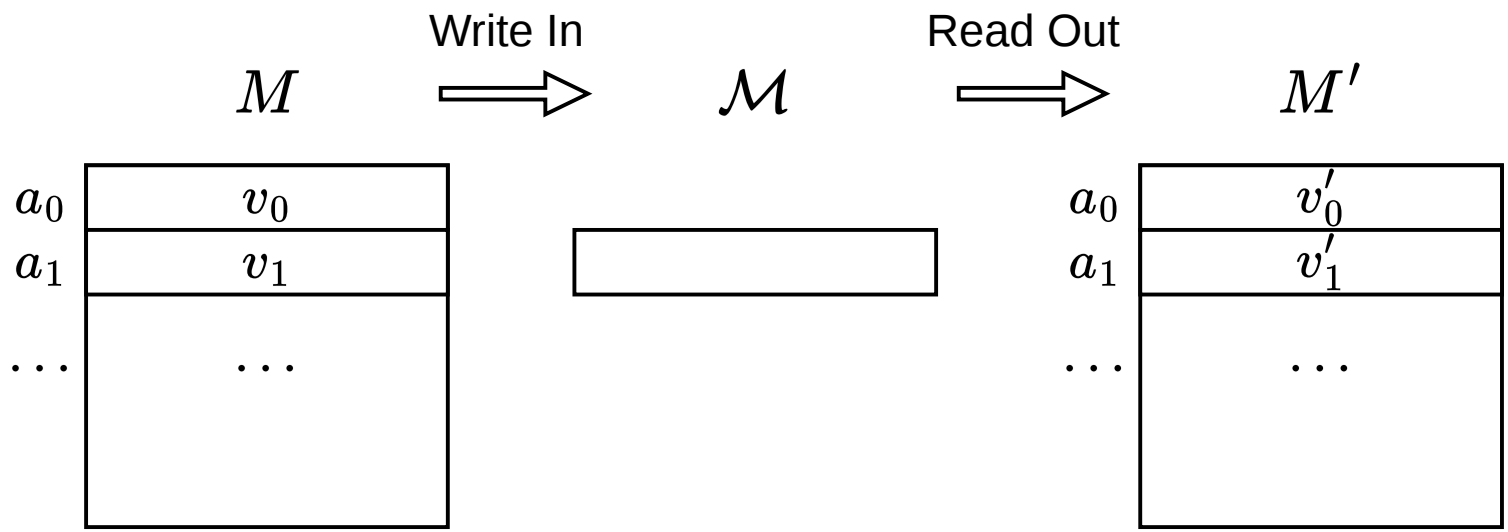


$$M \stackrel{?}{=} M'$$

$$\frac{1}{|M|}\sum_{m=1}^{|M|}\mathbb{1}\left[\text{clean-up}(a^m \circledast \mathcal{M}) = v^m\right]$$

Memory Capacity Test



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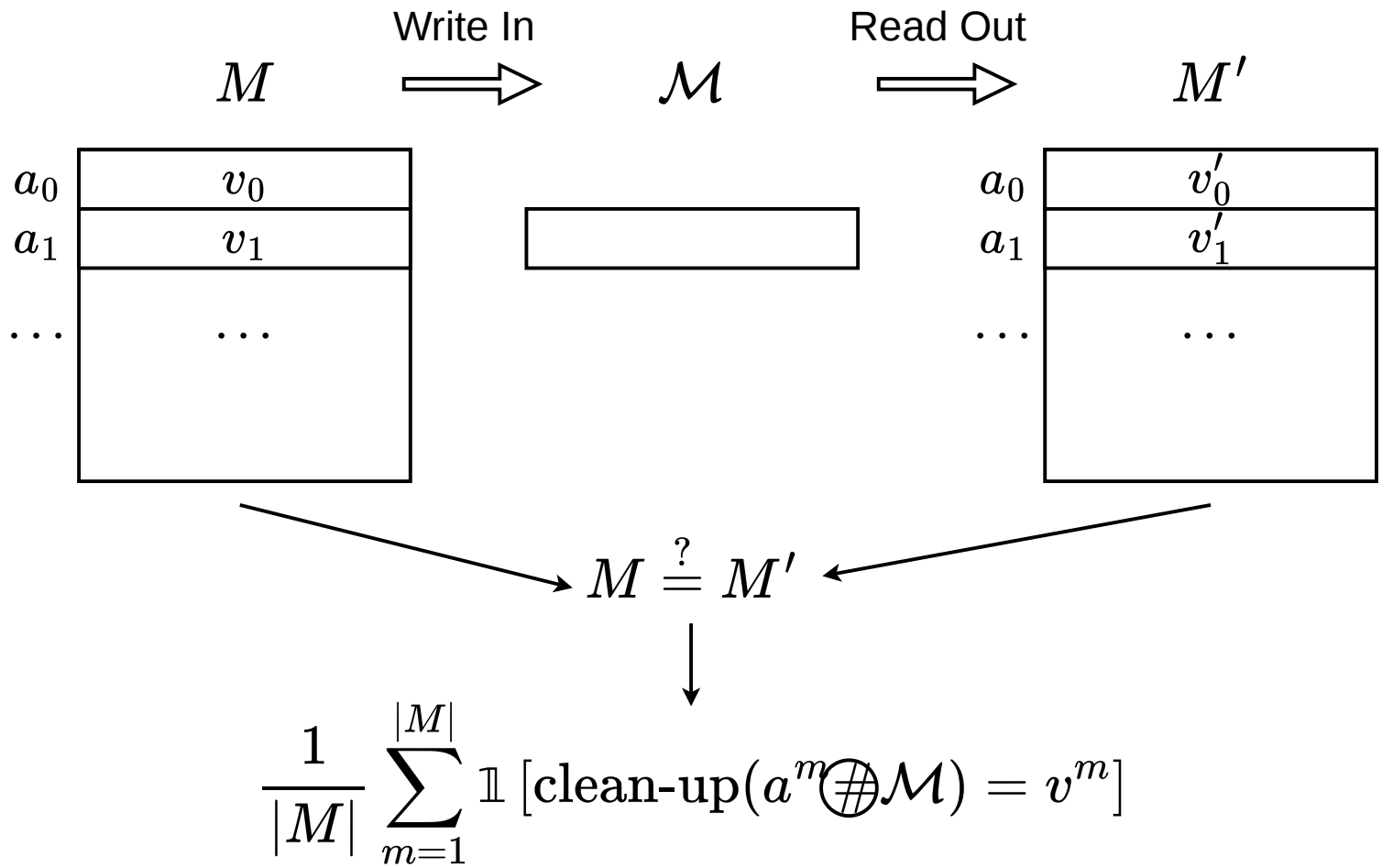


Memory Capacity Result

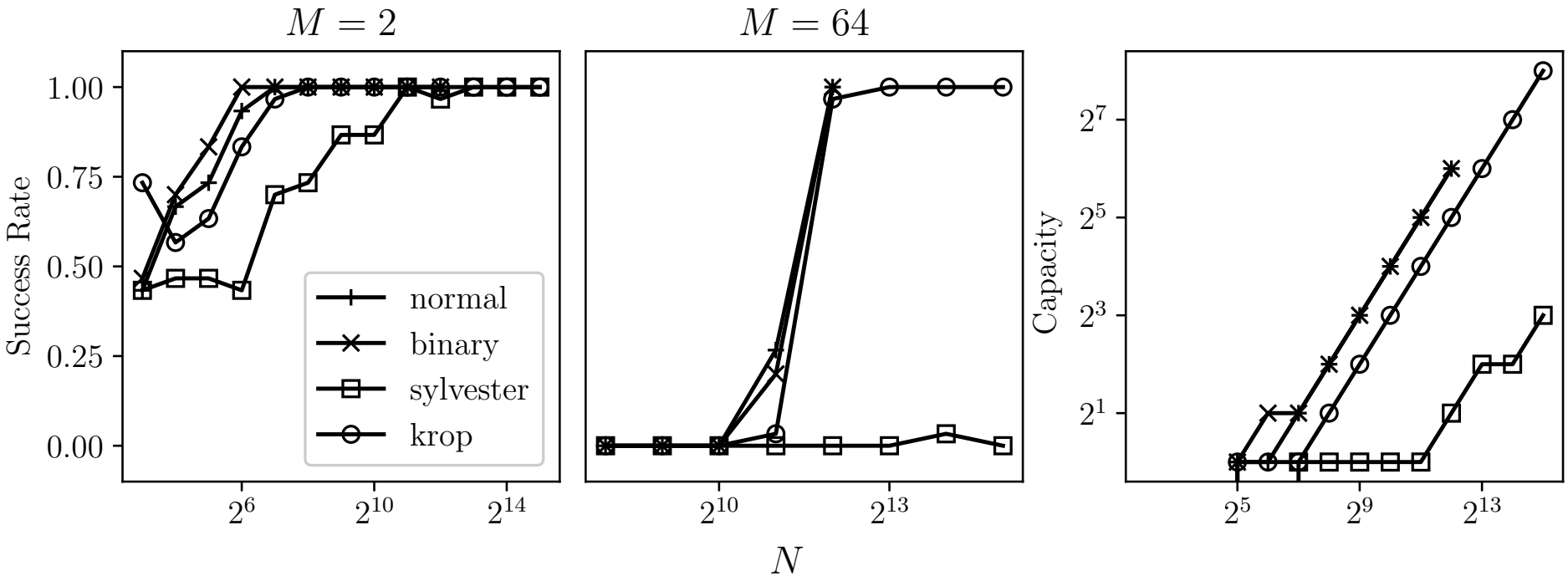
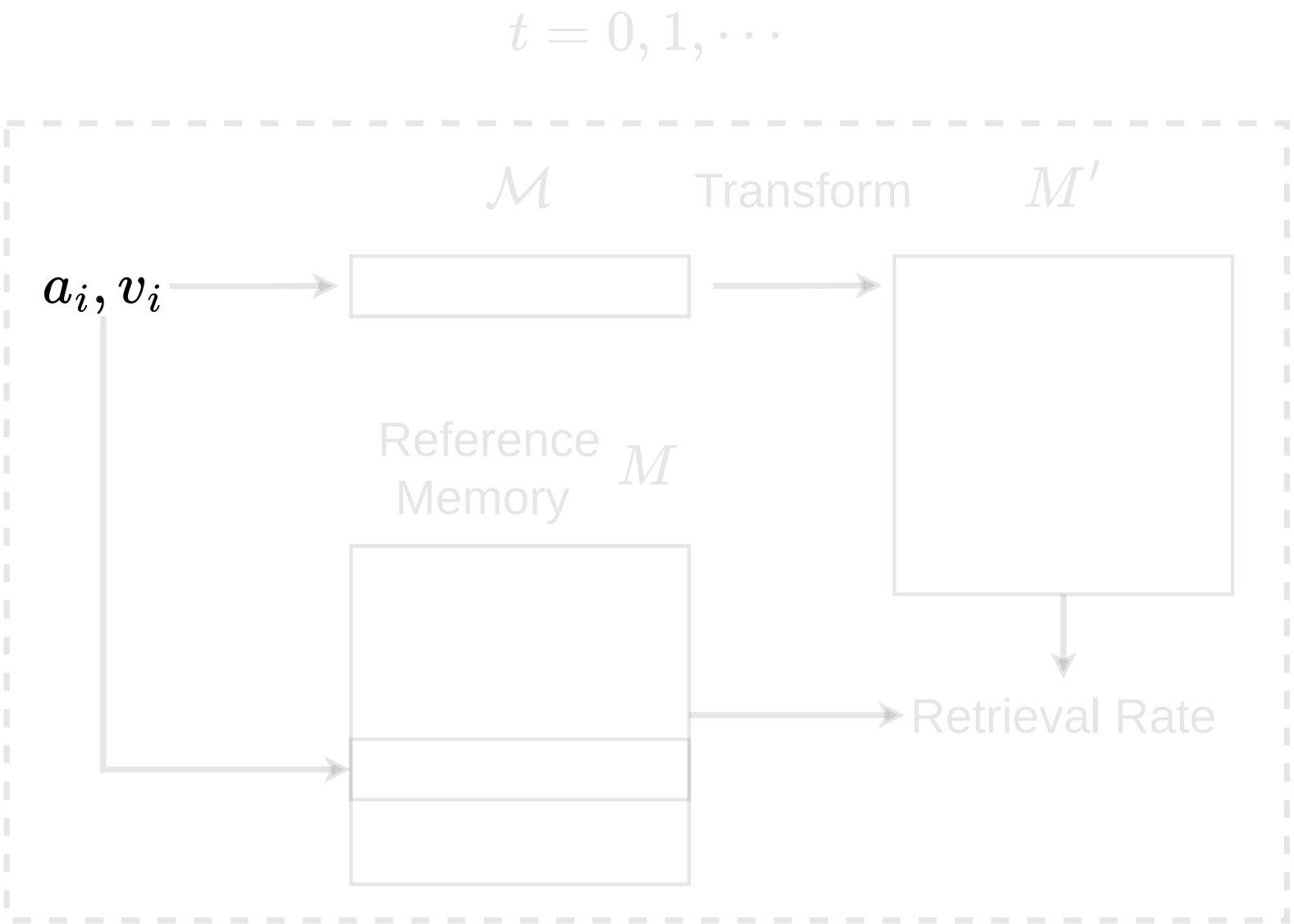


Figure: Success rates (left) and memory capacity (right) for each kind of value codebook.

Mutable Key-Value Memory Test

Does read-out error accumulate **over time**?

Mutable Key-Value Memory



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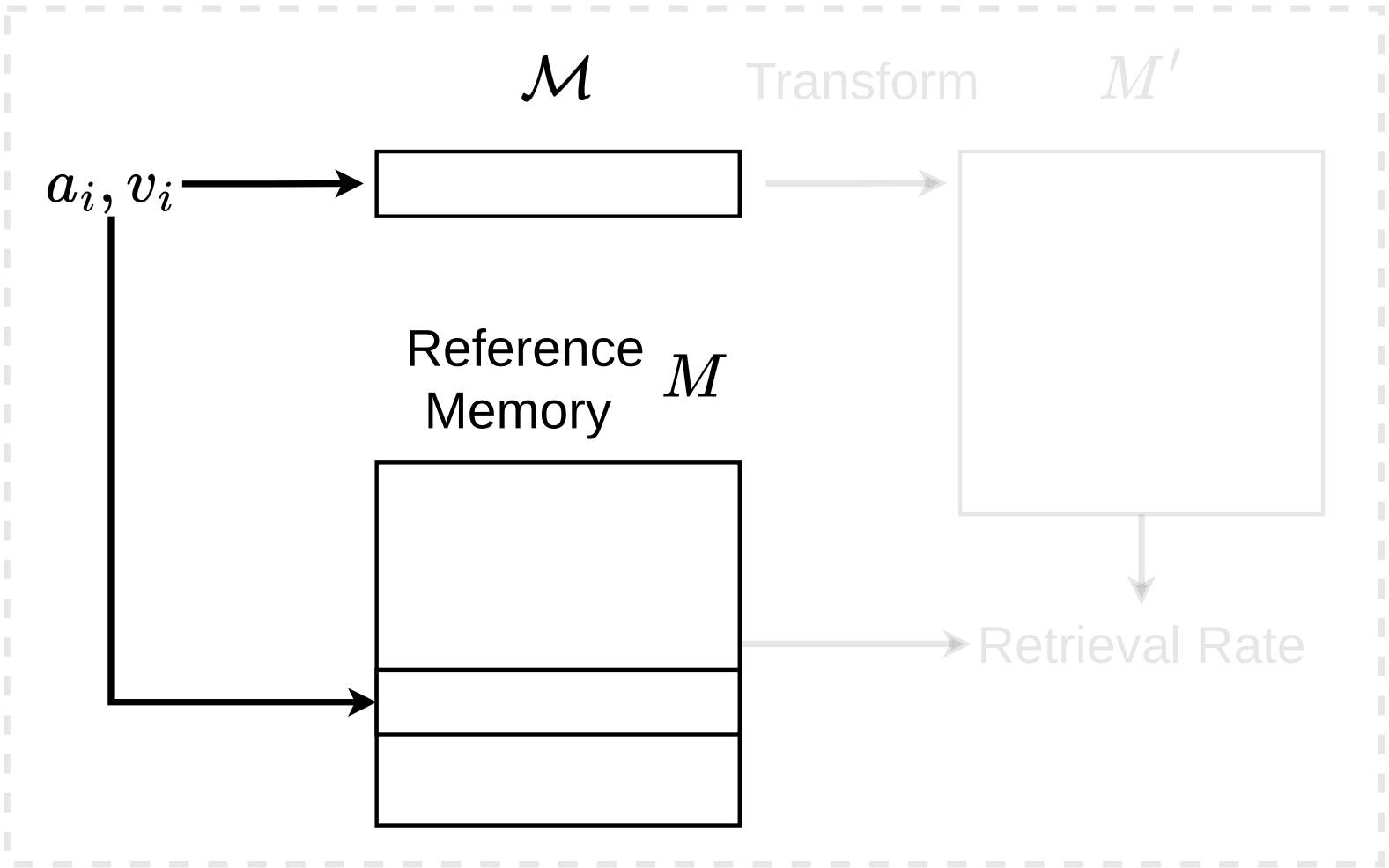
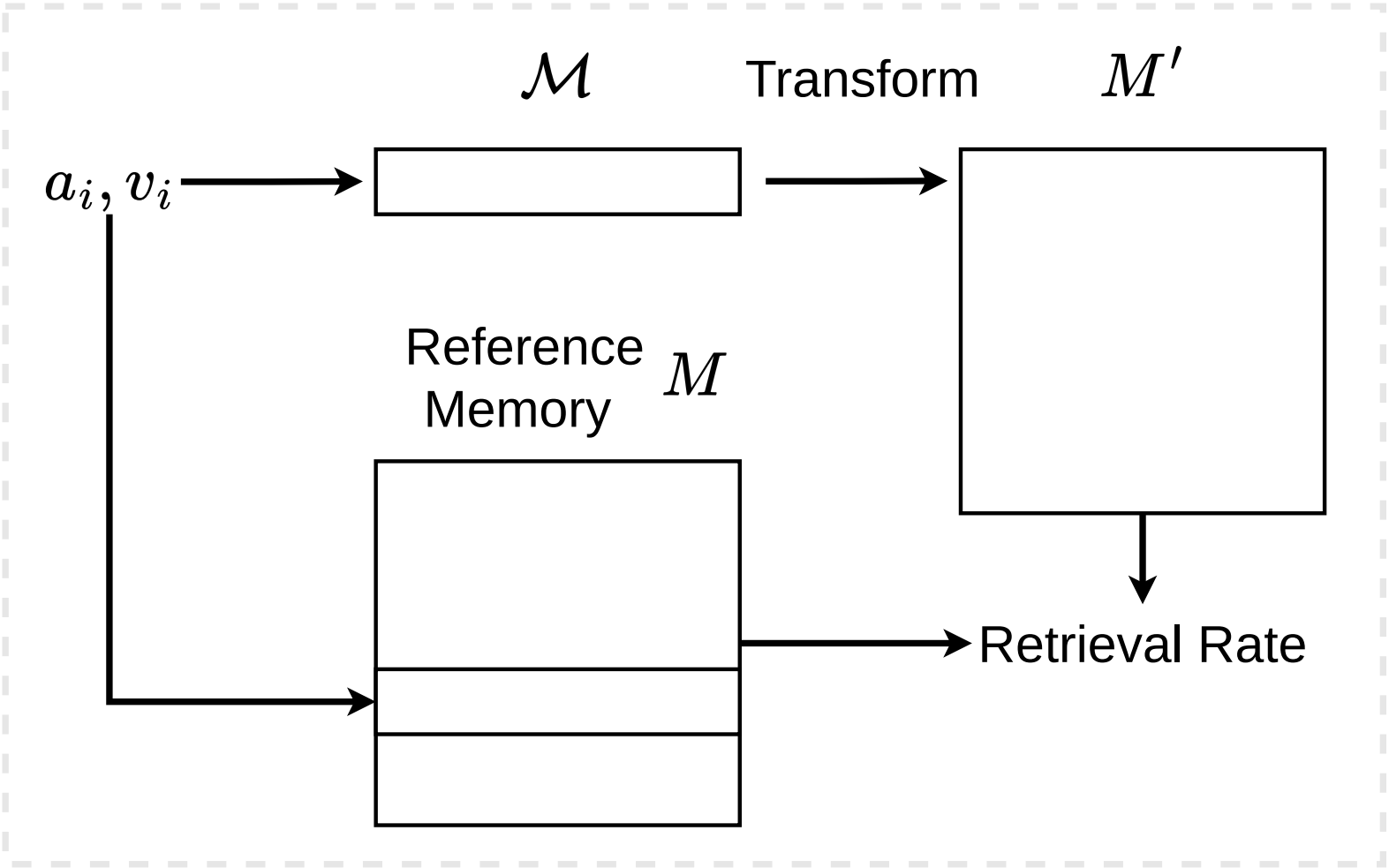


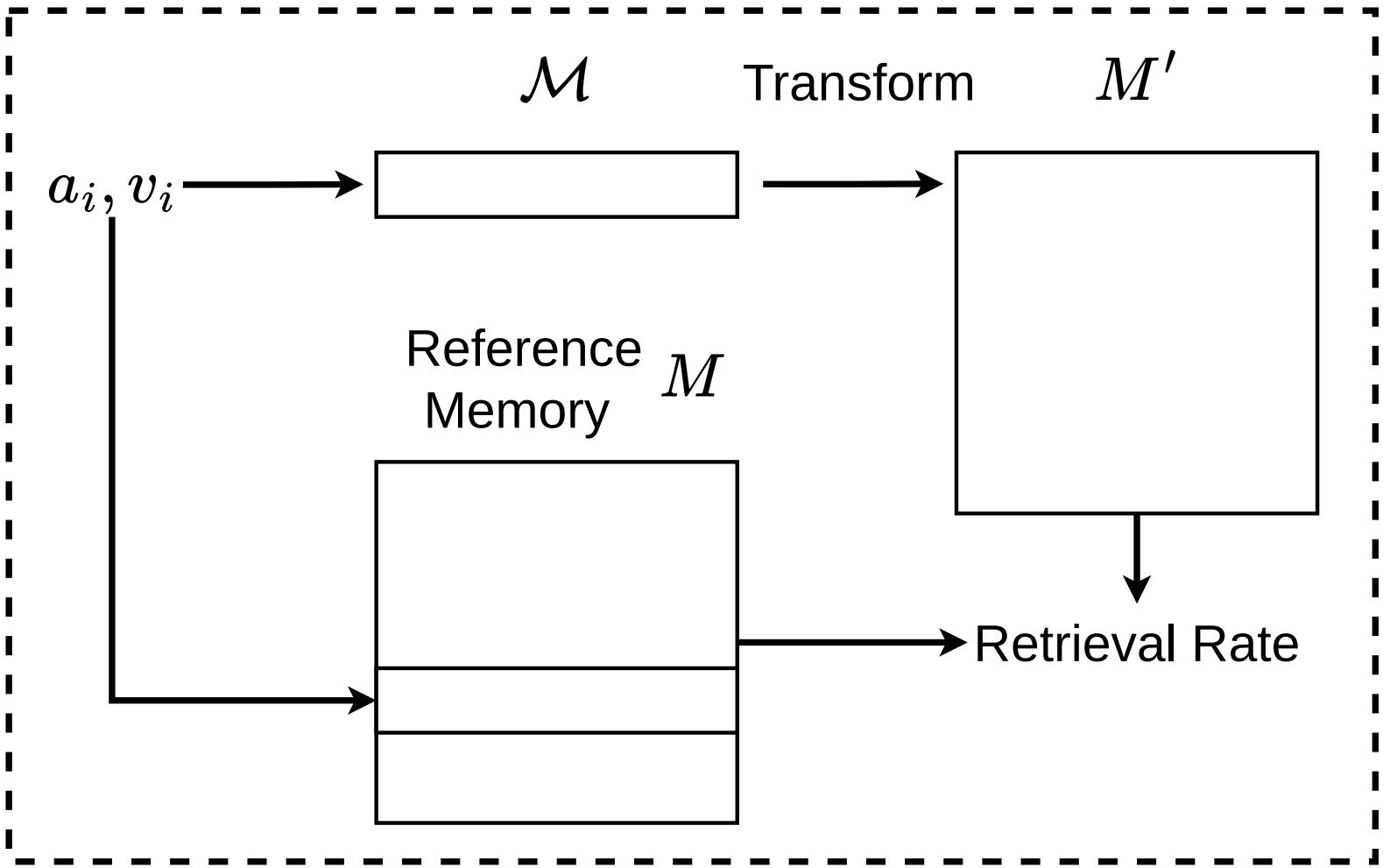
Mutable Key-Value Memory

$t = 0, 1, \cdots$



Mutable Key-Value Memory

$$t = 0, 1, \cdots$$



Mutable Key-Value Memory Result

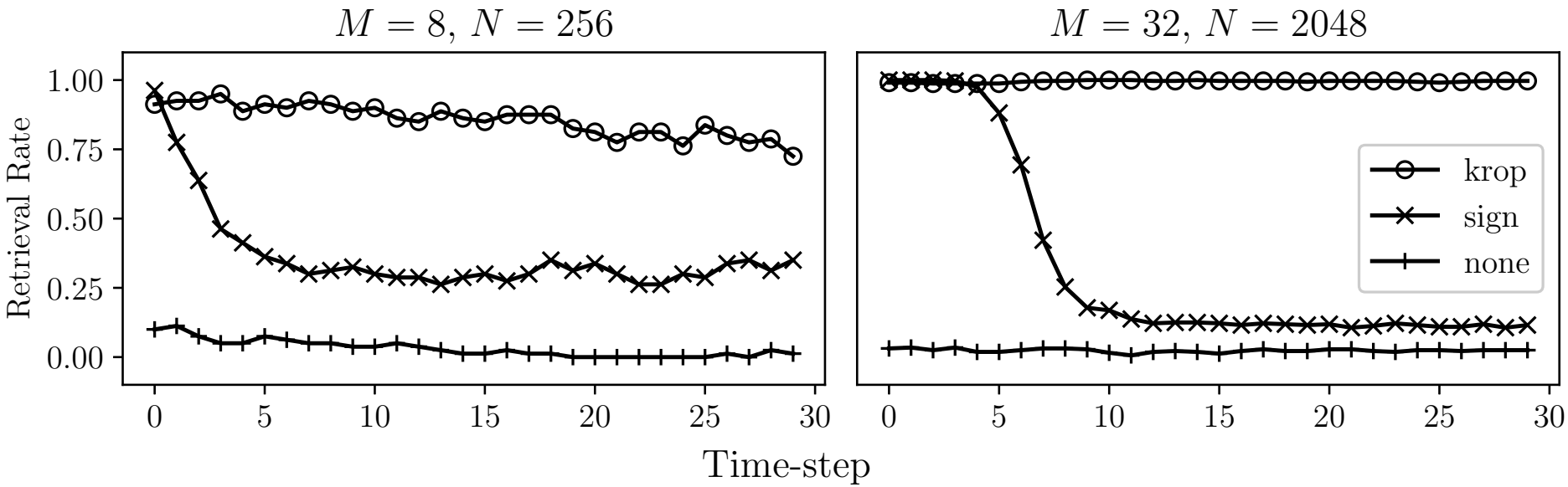


Figure: Examples of mutable VSA memory retrieval rates by time-step.

Mutable Key-Value Memory Result

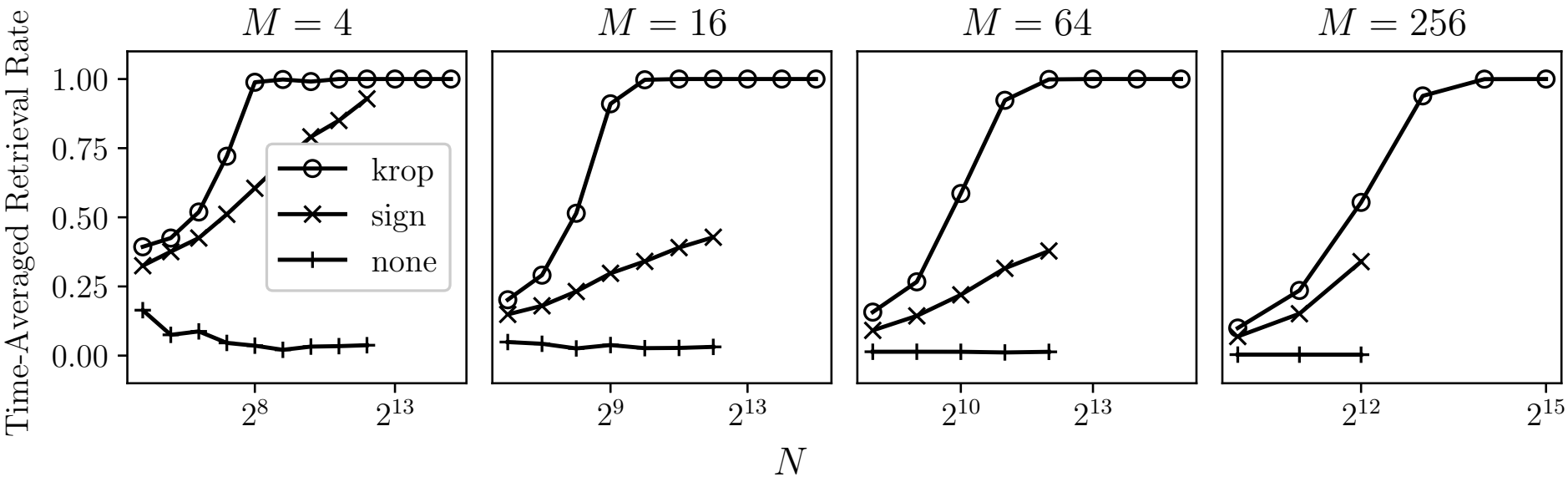
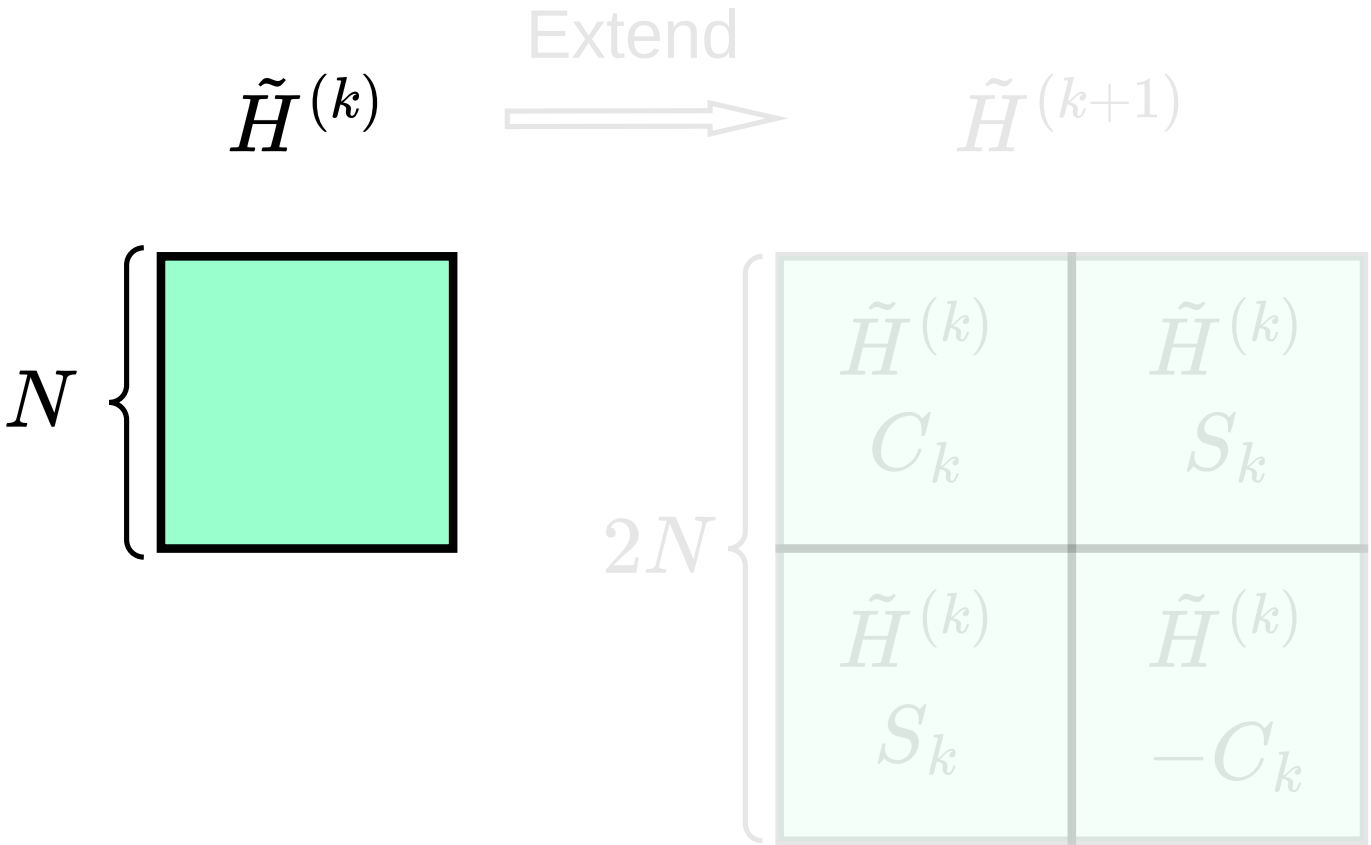


Figure: Mutable VSA memory retrieval rates averaged over 10 independent trials and 30 read-write steps.

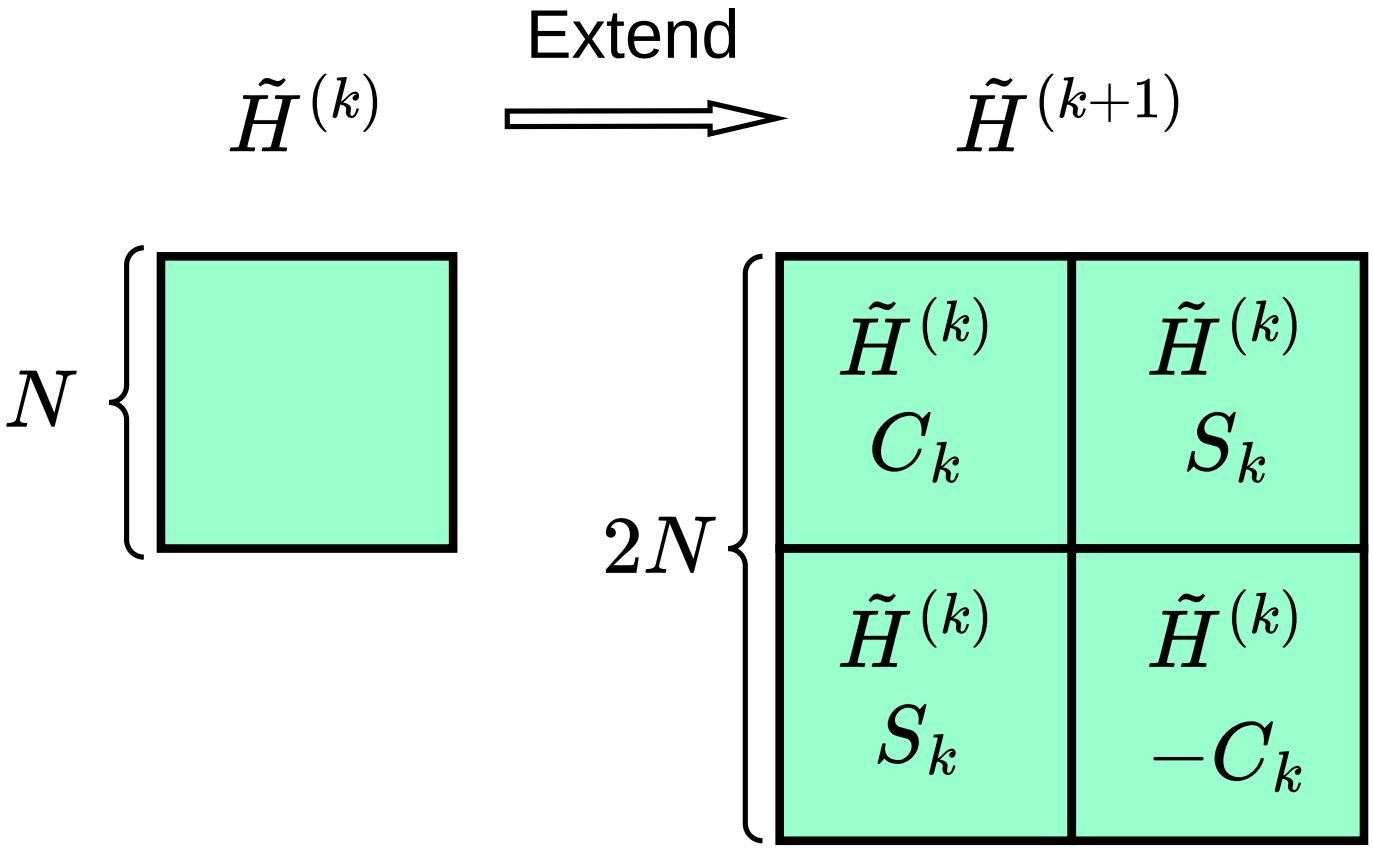
Limitation

`krop` improves clean-up efficiency, but...

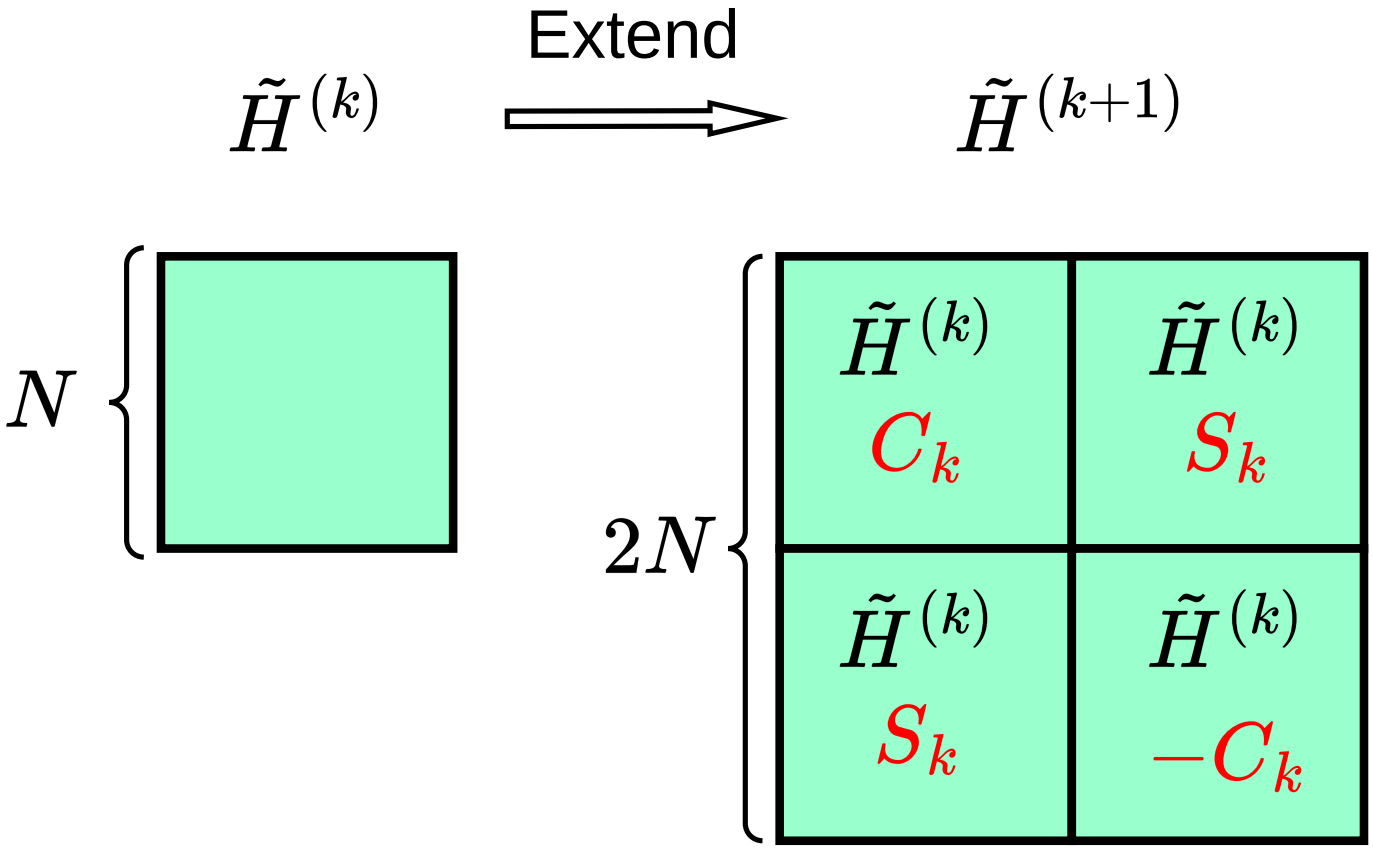
Distributional Miss Match



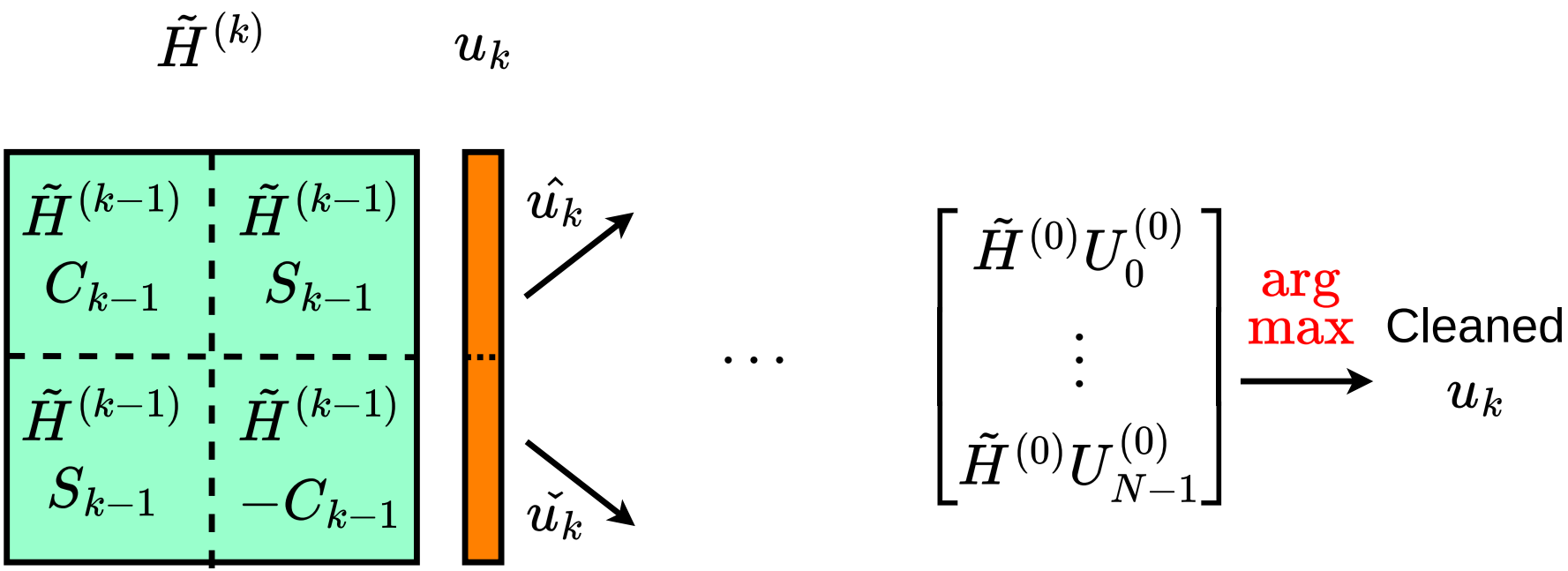
Distributional Miss Match



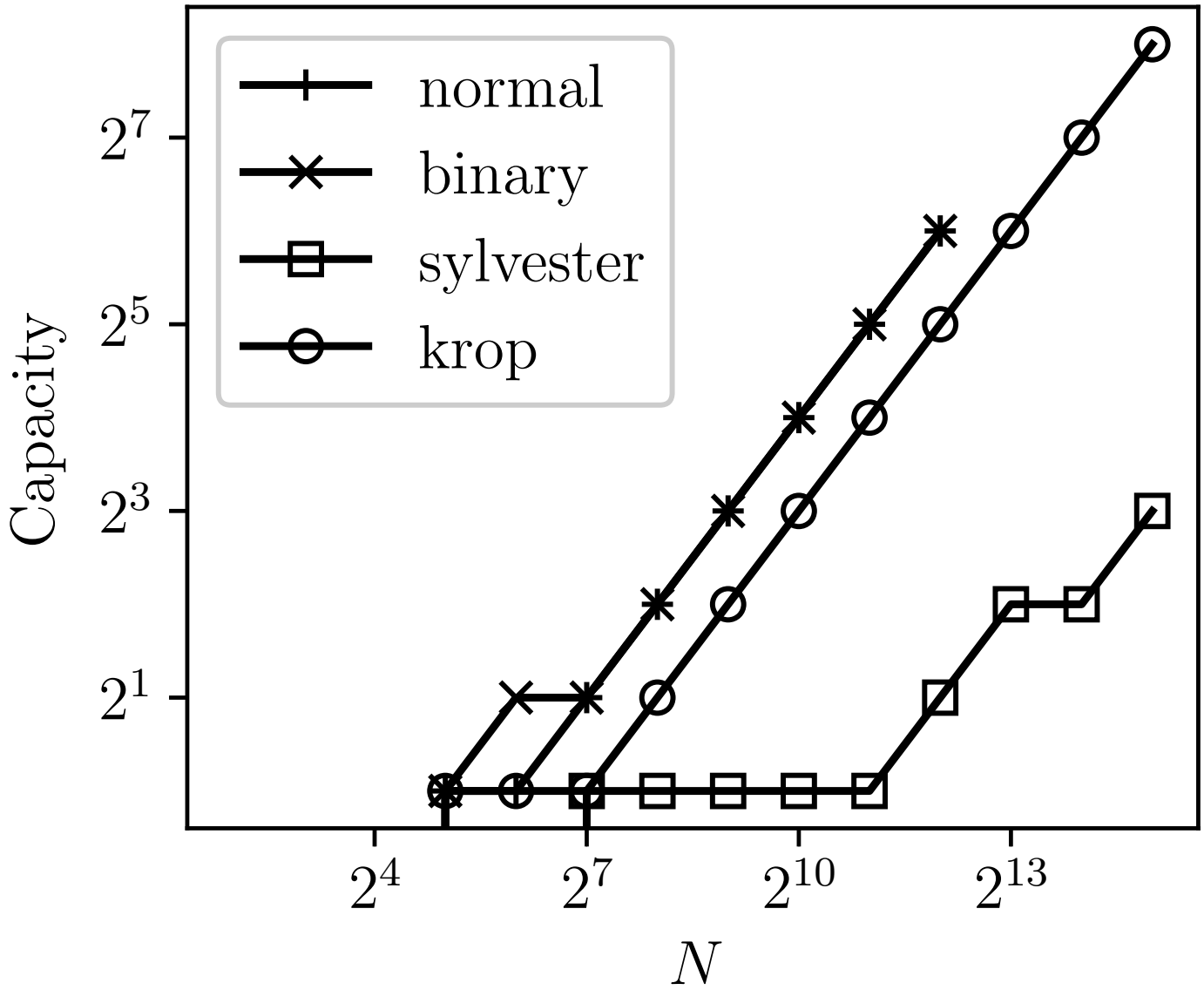
Distributional Miss Match



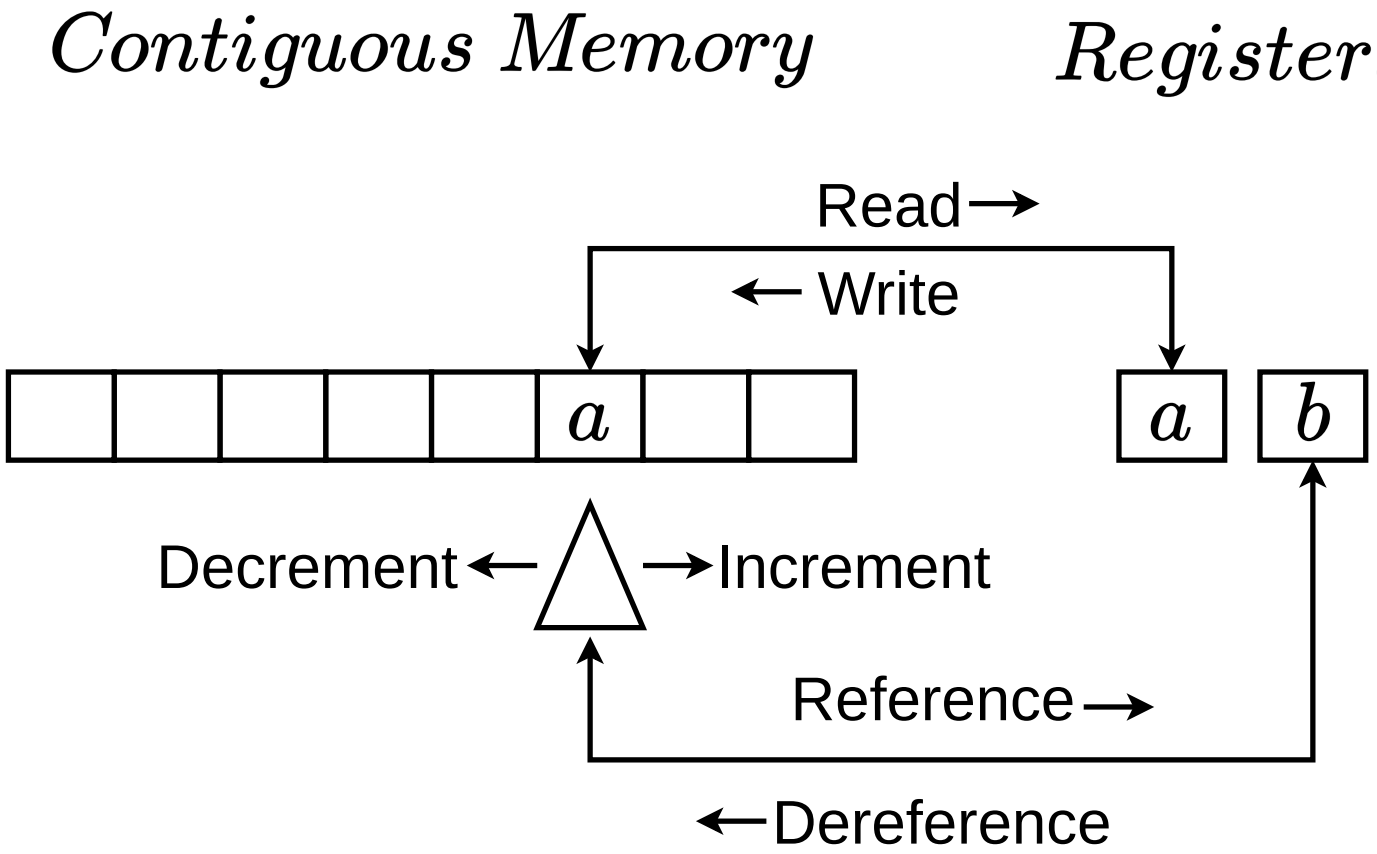
Non-differentiability



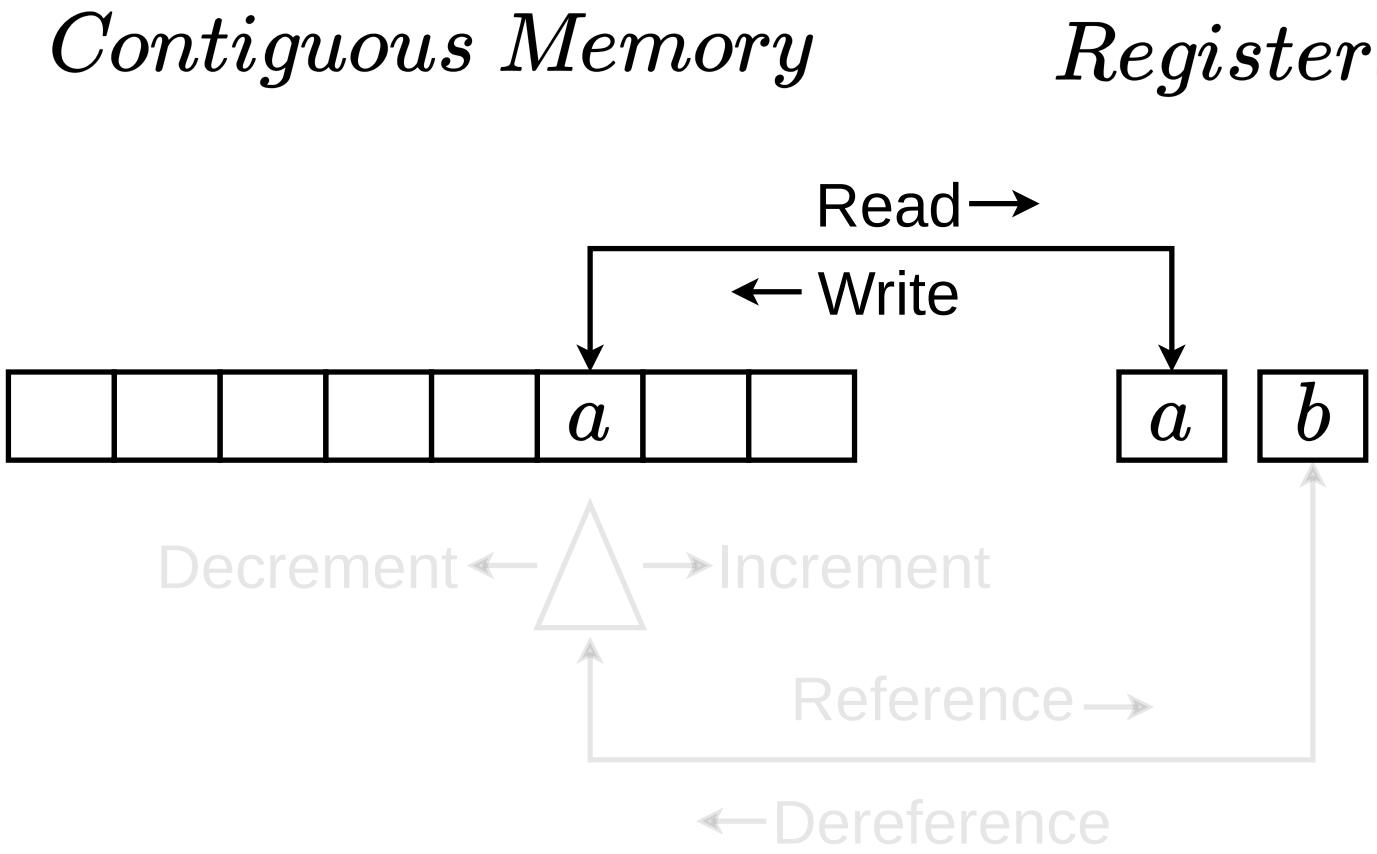
Required Large Dimension



Pointer Issue



Pointer Issue



Conclusion

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A codebook with linearithmic clean-up for VSA key-value memory.

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Compact codebook storage and efficient vector reconstruction.

Conclusion

`krop` is:

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It provides:

Compact codebook storage and efficient vector reconstruction.

Remaining limitations:

Distribution mismatch, non-differentiability, high-dimensional requirements and typing issue.

Thank You

Questions and Discussion

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Garrett Katz

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